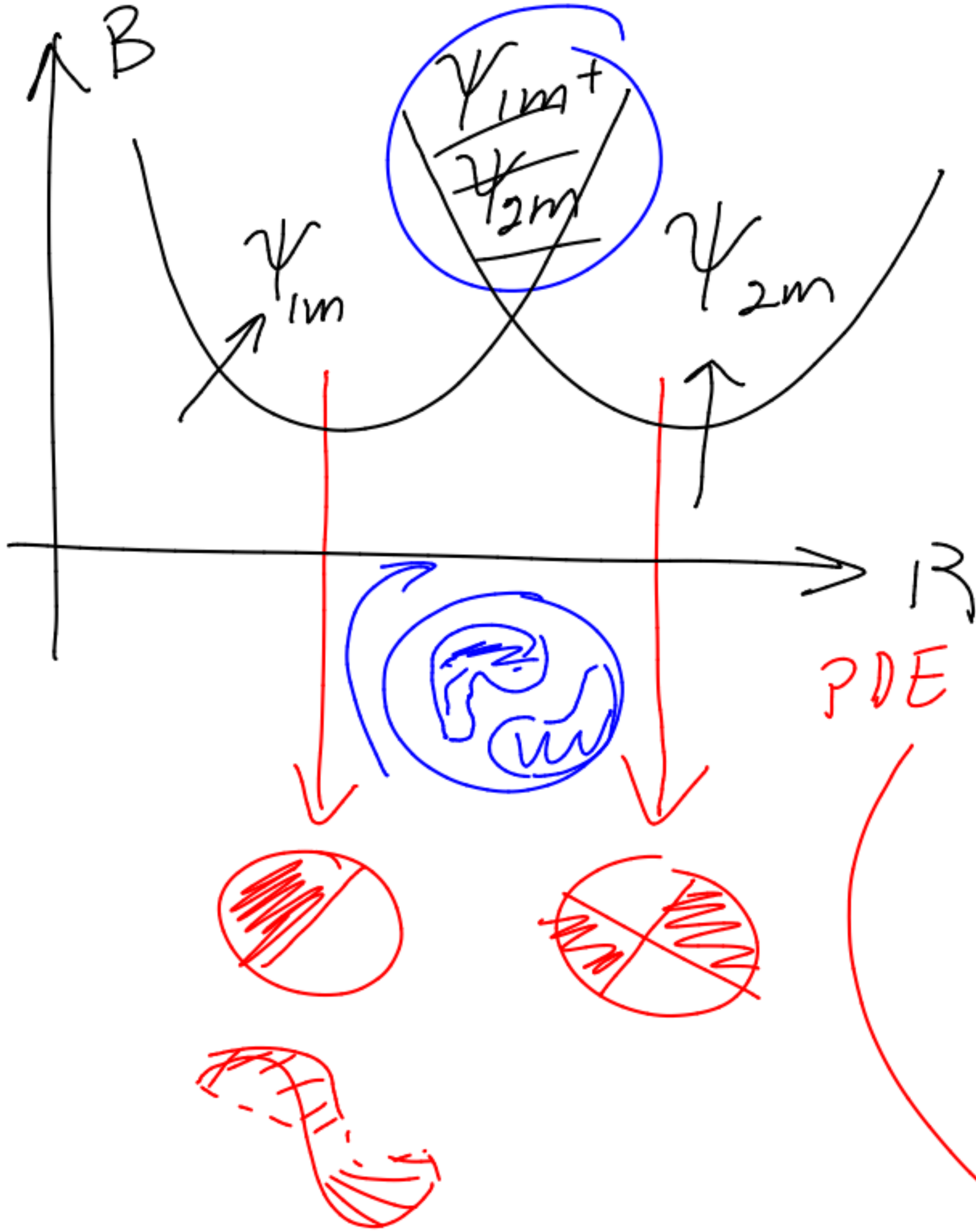
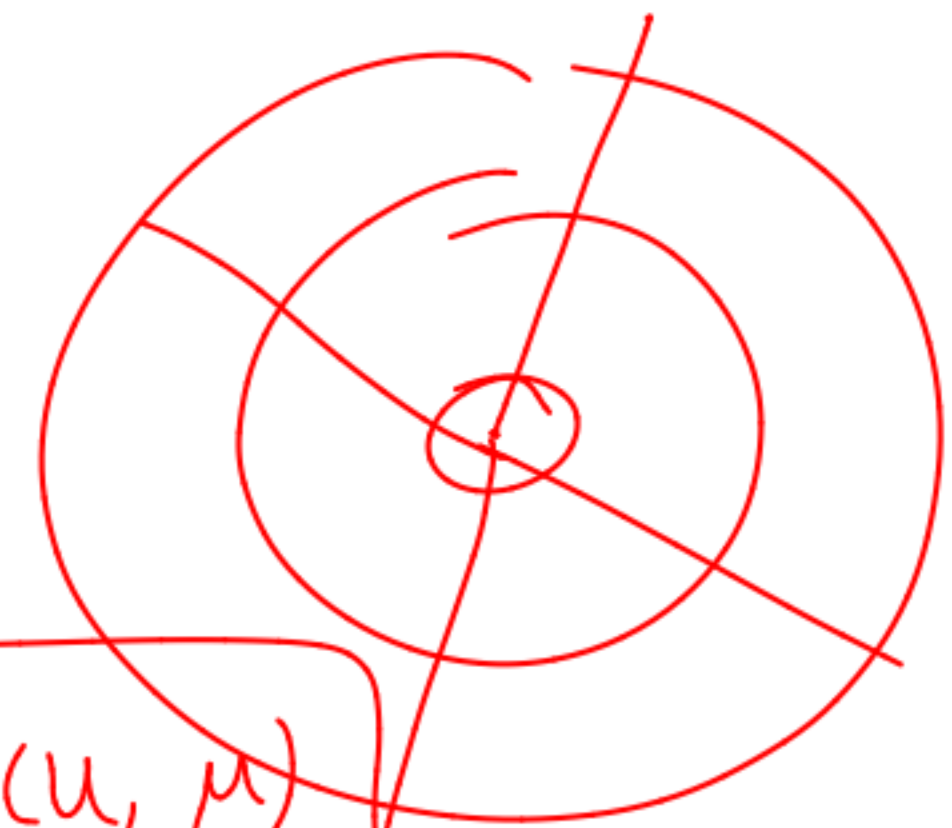




PDE $\boxed{\frac{\partial u}{\partial t} = F(u, \mu)}$

$$\frac{\partial u(r,\theta)}{\partial t} = \cdots \frac{1}{r^2} + \cdots \frac{1}{r^4}$$

$$u(r,\theta) = \sum_{n=1}^{N} z_n(t)\, \psi_{nm}(r,\theta) \text{ H.c.}$$

$$\Downarrow$$

$$\boxed{\frac{d z_n}{dt} = f(z_1, \dots, z_N)} \text{ ODE}$$

Galerkin Projection

Given: $\frac{\partial u}{\partial t} = F(u, \lambda)$ (PDE)

Orthogonality

$$u(x,t) = \sum_{k=1}^{N} \underbrace{a_k(t)}_{\text{amplitude}} \underbrace{\phi_k(x)}_{\text{spatial modes}}, \quad \langle \phi_i, \phi_j \rangle = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

Want: $\frac{da_k}{dt} = f(a_1, \ldots, a_n)$ (ODE)

Substituting sol. $u(x,t)$ into (PDE)

$$\sum_{k=1}^{N} \frac{da_k}{dt} \phi_k(x) = F\left(\sum_{k=1}^{N} a_k \phi_k, \lambda\right)$$

$$\left\langle \sum_{k=1}^{N} \frac{da_k}{dt} \phi_k(x), \phi_j(x) \right\rangle = \left\langle F\left(\sum_{k=1}^{n} a_k \phi_k\right), \phi_j \right\rangle$$

$$\sum_{k=1}^{N} \frac{da_k}{dt} \underbrace{\langle \phi_k , \phi_j \rangle}_{\substack{=1,\ k=j \\ =0,\ k\neq j}} = \left\langle F\left(\sum a_k \phi_k\right), \phi_j \right\rangle$$

ODE

$$\boxed{\frac{da_j}{dt} = \left\langle F\left(\sum_{k=1}^{N} a_k \phi_k\right), \phi_j \right\rangle}, \quad j = 1, \dots, N$$

$$u^3 \longrightarrow (a_1\phi_1 + a_2\phi_2 + a_3\phi_3)^3$$

- One can use other techniques (Proper Orthogonal Decomposition) to identify spatial modes directly from a data set.

- Dynamical Sys Theory (equilibria & stability for 2D systems)

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

- Bifurcation Theory
  - Saddle-node bif.
  - Transcritical
  - Pitchfork
  - Hopf Bif.

Group Theory

Ch 2 Bifurcation Theory

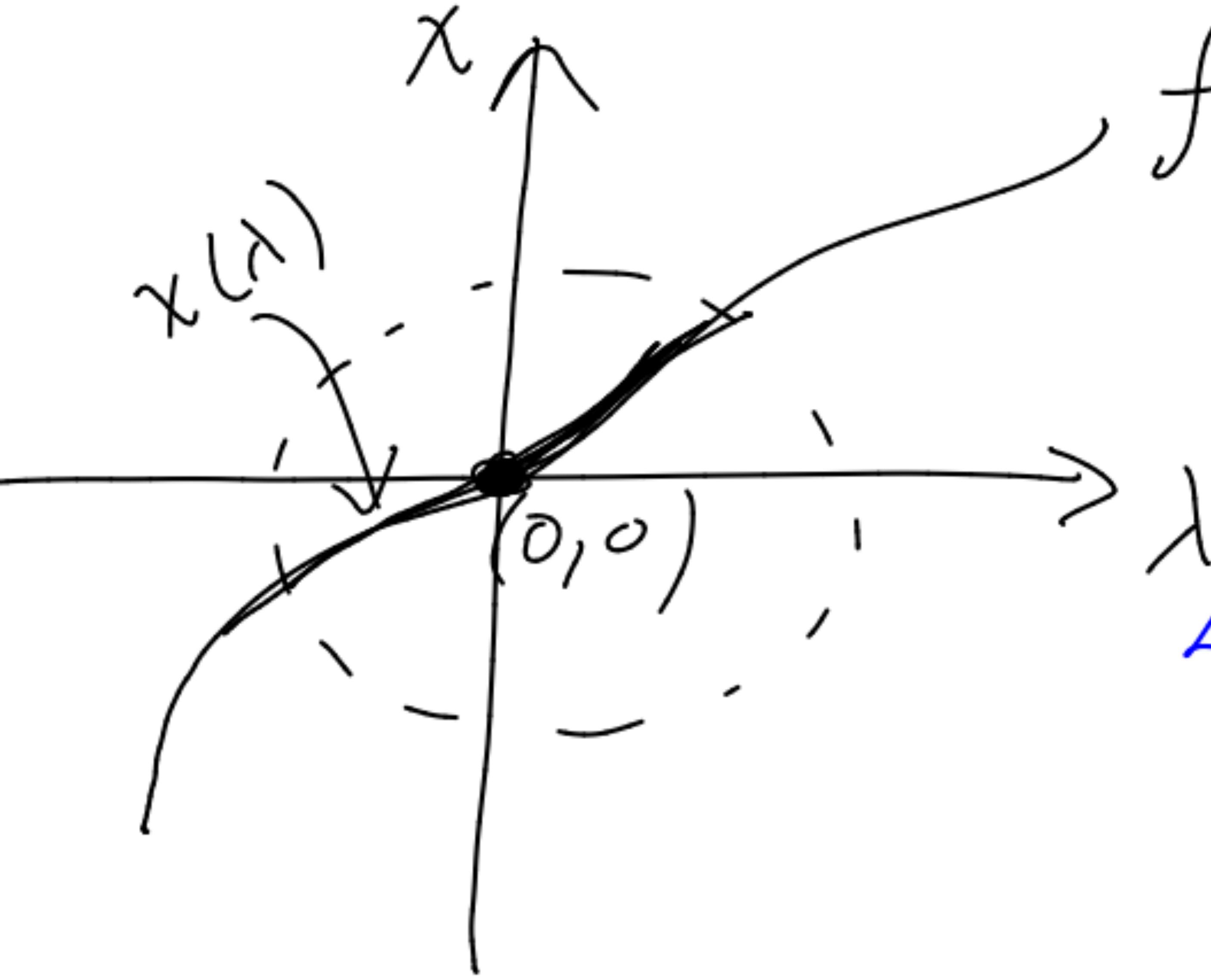
Bif. Theory is the study of how systems change behavior upon changes of parameters.

Want to solve: $f(x, \lambda) = 0$

Implicit Function Theorem $\}$ Assume $f(0,0) = 0$

$$\frac{\partial f}{\partial x}(0,0) \neq 0$$

Then there exists a smooth sol. $x(\lambda)$, i.e. parametrized by λ, s.t. $f(x(\lambda),$



$f(x,\lambda)$ ⊙ A bif. requires the IFT to fail. !

Local Bif. Condition

$$\boxed{\begin{aligned} f(x_c,\lambda_c) &= 0 \\ \left.\frac{\partial f}{\partial x}\right|_{(x_c,\lambda_c)=(0,0)} &= 0 \end{aligned}}$$

Saddle-Node Bif.

Assume: $\dot{x} = \lambda - x^2 + h.o.t.$; $f(x,\lambda) = \lambda - x^2$

$f_x = -2x$

S1 Equil: $\lambda = 0,\ x = 0$

S2 Bif. Cond. $f(0,0) = 0$ ✓

$f_x(0,0) = -2(0) = 0$ ✓

Equil: $x = \pm\sqrt{\lambda}$, only exists when $\lambda \geq 0$

S3 Stability

$f_x(x_+ = +\sqrt{\lambda}) = -2\sqrt{\lambda} < 0 \Rightarrow x_+ = +\sqrt{\lambda}$ is stable

$f_x(x_- = -\sqrt{\lambda}) = +2\sqrt{\lambda} > 0 \Rightarrow x_- = -\sqrt{\lambda}$ is unstable

S4 Bif. Diagram