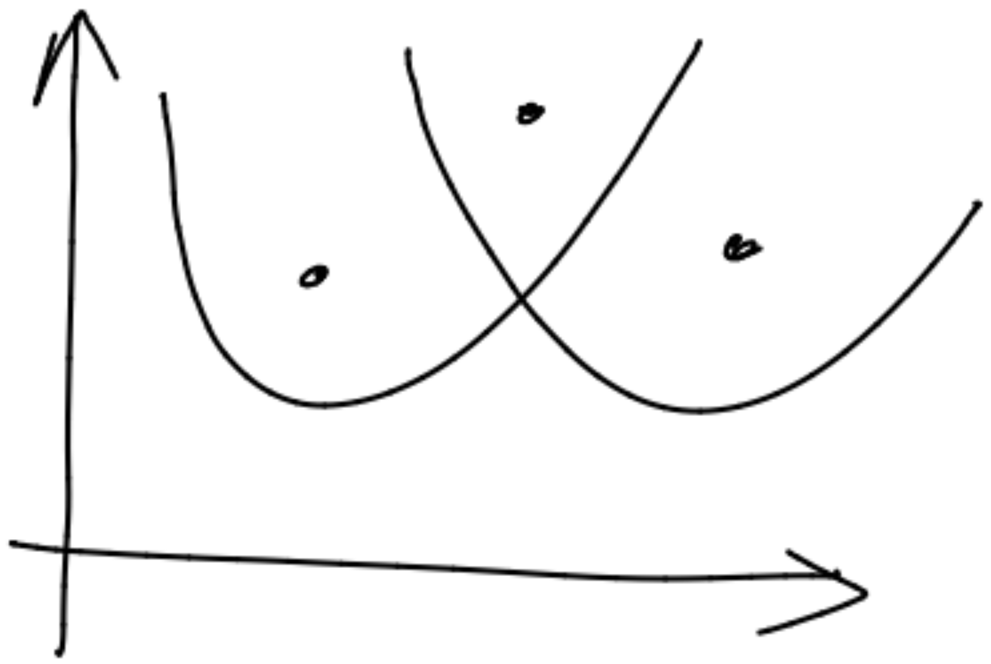
Homework 2: $u = \dfrac{\partial^2 u}{\partial x^2} = 0, \quad x = 0, L$



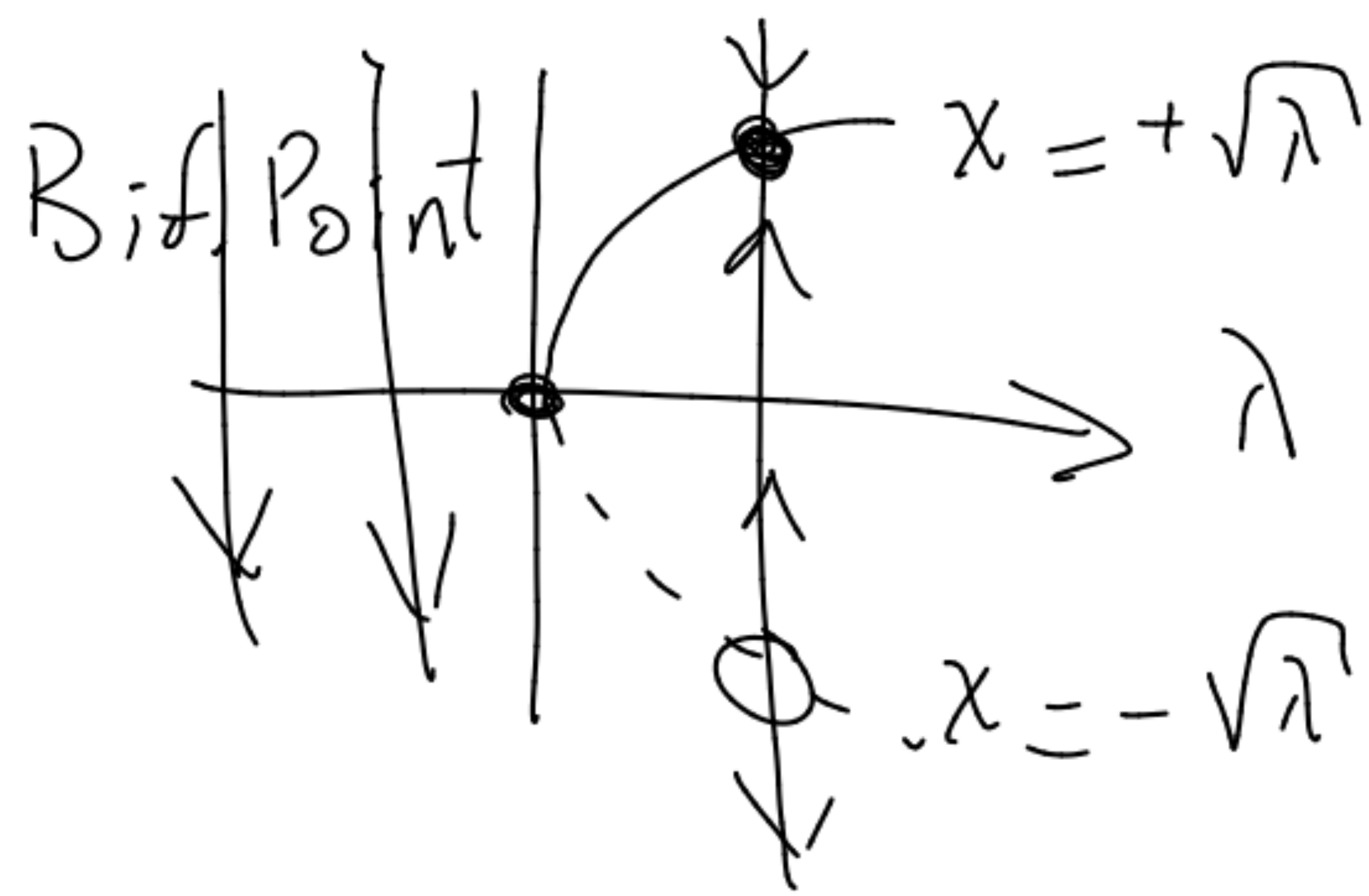
$$u = \sum G(t)\,\phi(x)$$

__Bif. Cond__

$\dot{x} = f(x, \lambda)$

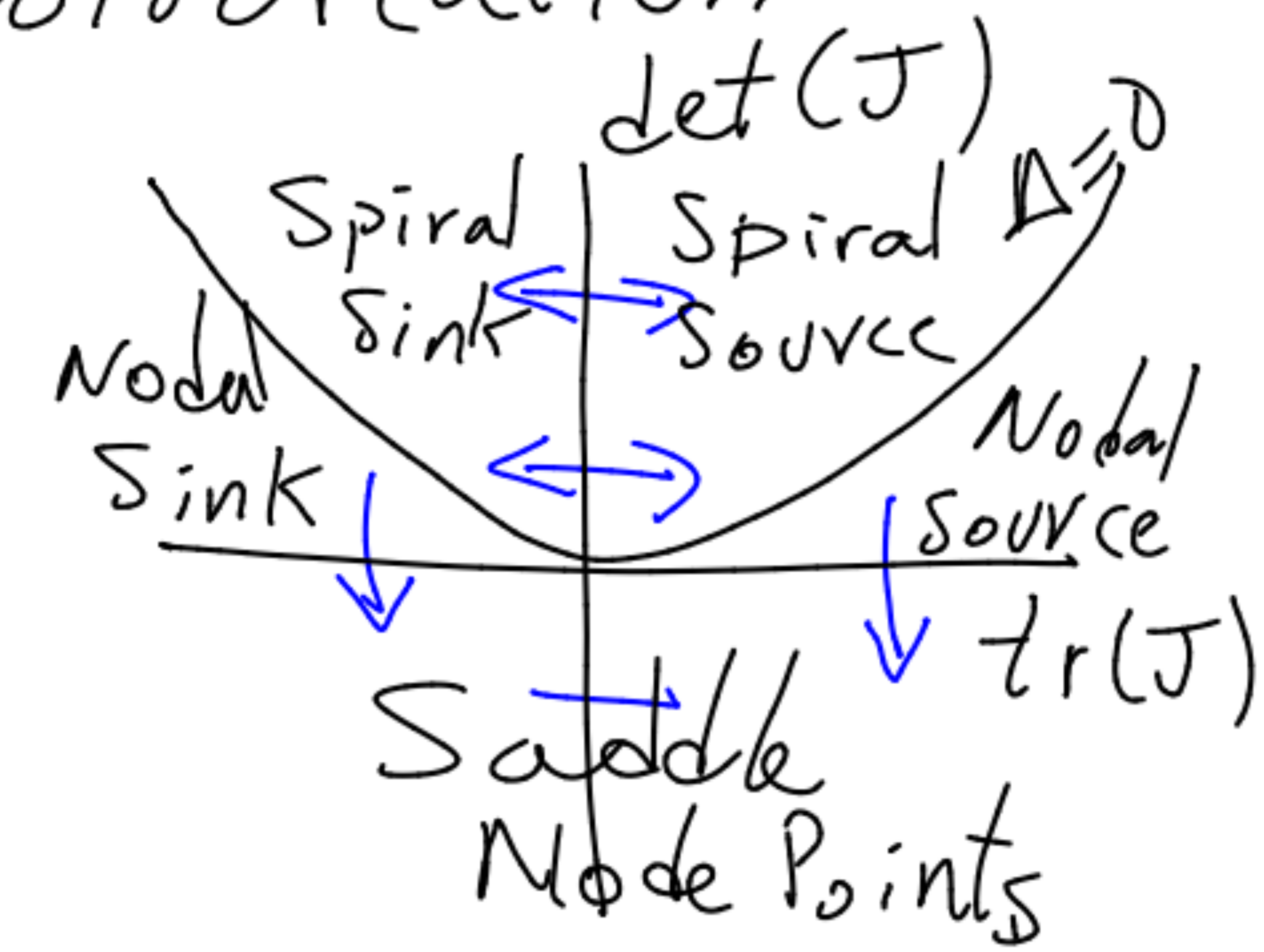
$$\boxed{\begin{array}{l} f(0,0) = 0 \text{ — equil. cond.} \\ f_x(0,0) = 0 \text{ — Failure of IFT} \end{array}}$$

e.g.) $\dot{x} = \lambda - x^2 = f(x, \lambda)$



Def. The codimension of a bifurcation is the difference between the dim. of the phase space and the dim. of the locus of the boundary (point, line, surface) that determines the bifurcation

e.g.)
$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$
OR
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{J} \begin{bmatrix} x \\ y \end{bmatrix}$$



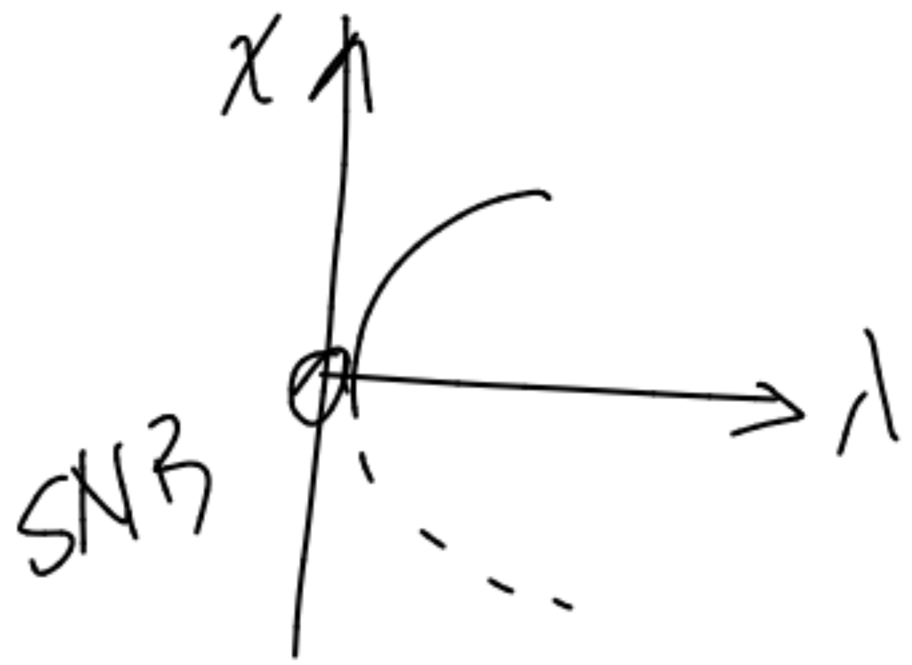
$$\sigma^2 - tr(J)\,\sigma + det(J) = 0$$

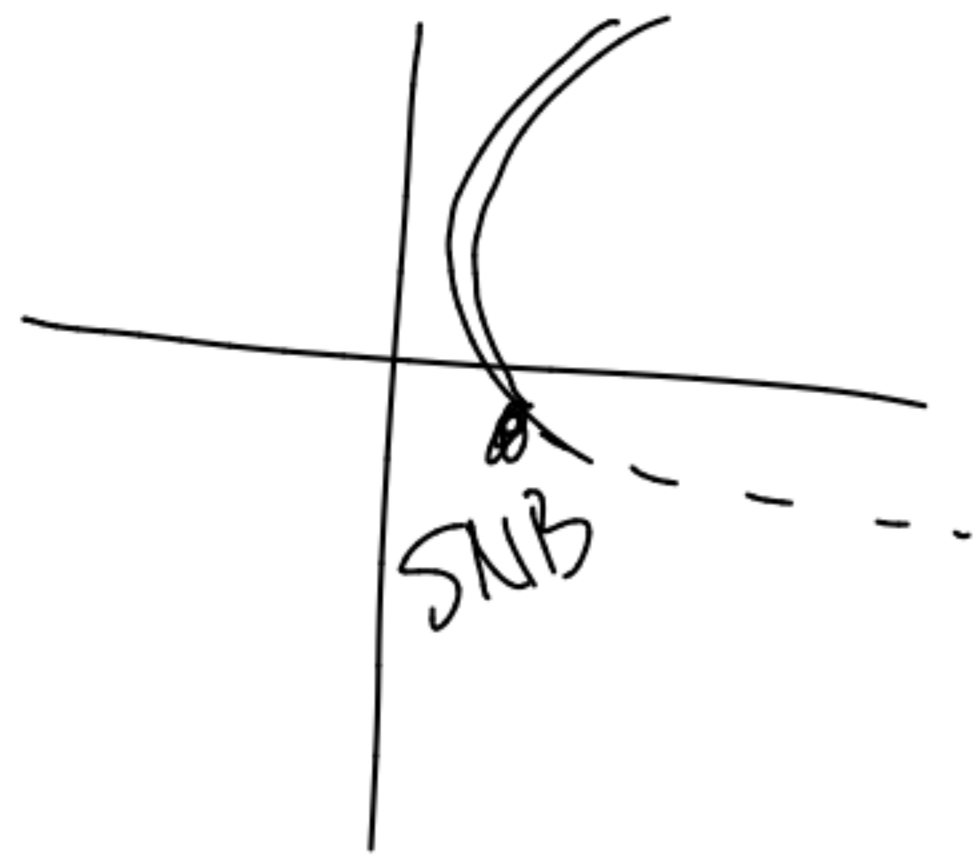
$$\sigma = \frac{1}{2} tr(J) \pm \frac{1}{2}\sqrt{\underbrace{tr^2(J) - 4 det J}_{\Delta}}$$

Genericity

$$\frac{dx}{dt} = f(x,\lambda) + \varepsilon p(x,\lambda)$$

$$\underset{\text{Taylor}}{=} \lambda - x^2 + \lambda\left(1 + \varepsilon v_1 + \varepsilon v_2 \lambda\right) + \varepsilon\left(v_3 + v_4 \lambda\right)x + \ldots \overset{\text{set}}{=} 0$$

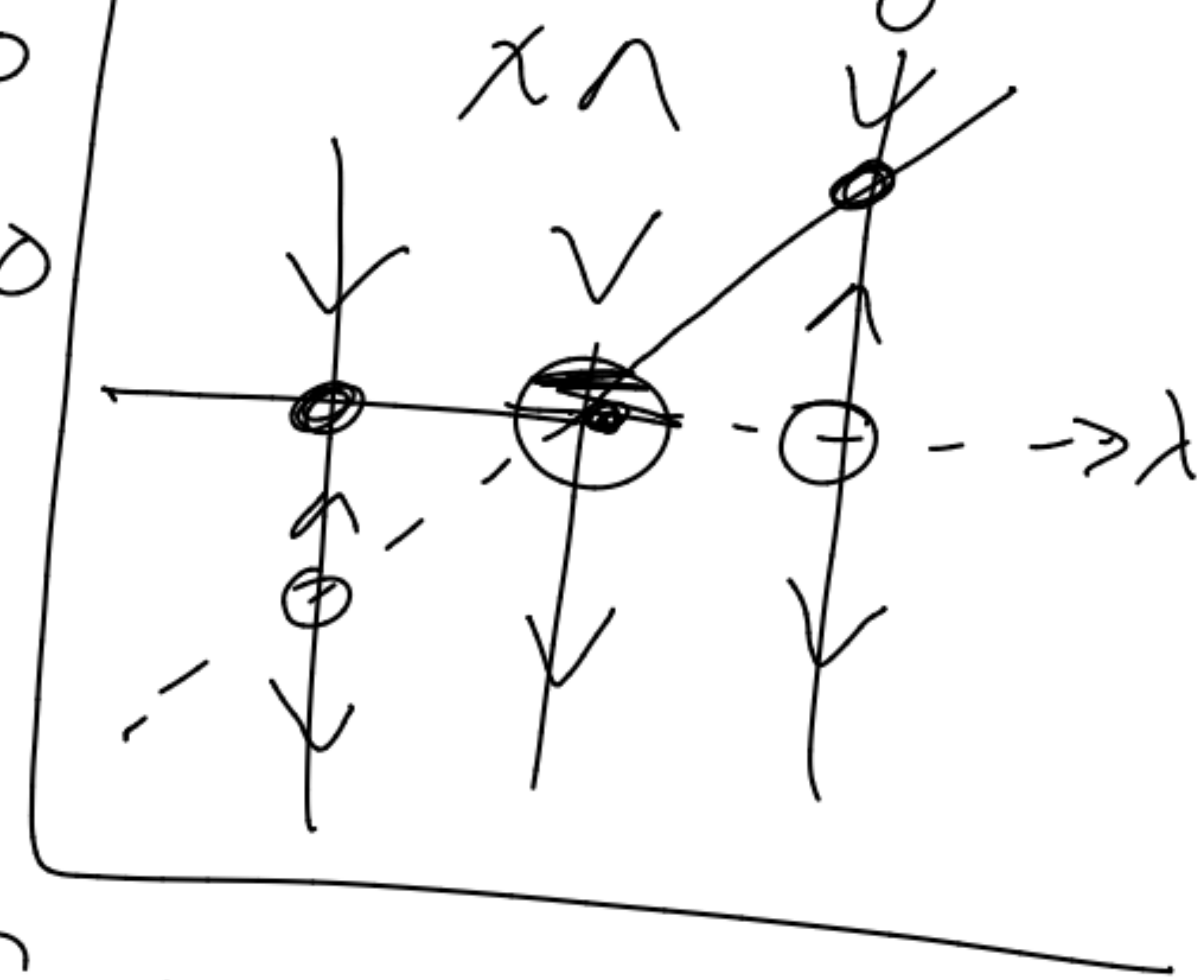




e.g.) Transcritical Bif.

$$\dot{x} = \lambda x - x^2 = f(x,\lambda) \quad \begin{cases} f(0,0)=0 \\ f_x(0,0)=0 \end{cases}$$

S3 Bif. Diagram



S1 Equilibria

$$(\lambda - x)x = 0 \Rightarrow x_1 = 0, \quad x_2 = \lambda$$

S2 Stability

$$f'(x) = \lambda - 2x$$

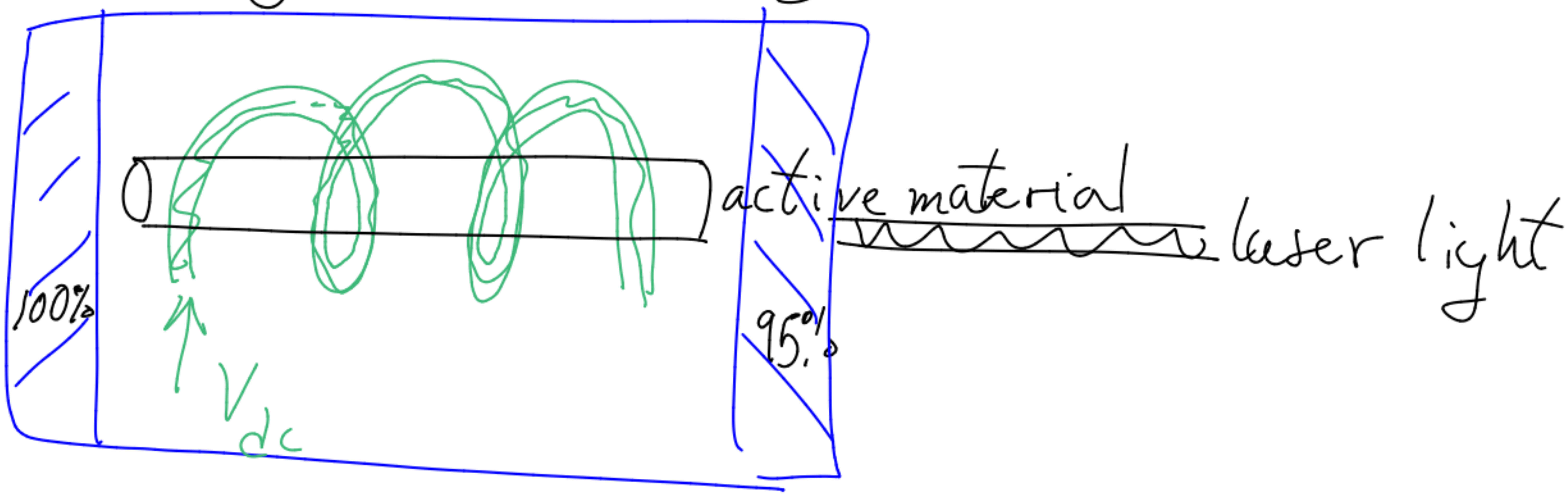
$$f'(x_1=0) = \lambda \Rightarrow x_1 = 0 \begin{cases} \text{stable if } \lambda<0 \\ \text{unstable if } \lambda>0 \end{cases}$$

$$f'(x_2=\lambda) = -\lambda \Rightarrow x_2 = \lambda \begin{cases} \text{stable if } \lambda>0 \\ \text{Unstable if } \lambda<0 \end{cases}$$

$f(0,\lambda) = 0 \Rightarrow$ genericity cond.

Application of Transcritical Bif.

Laser Dynamics (Ruby Laser)



Pitchfork Bif.

$$\dot{x} = \lambda x - x^3 = f(x,\lambda)$$

S1 Equilibria

$$(\lambda - x^2)x = 0 \Rightarrow x_1 = 0,\ x_{2,3} = \pm\sqrt{\lambda}$$

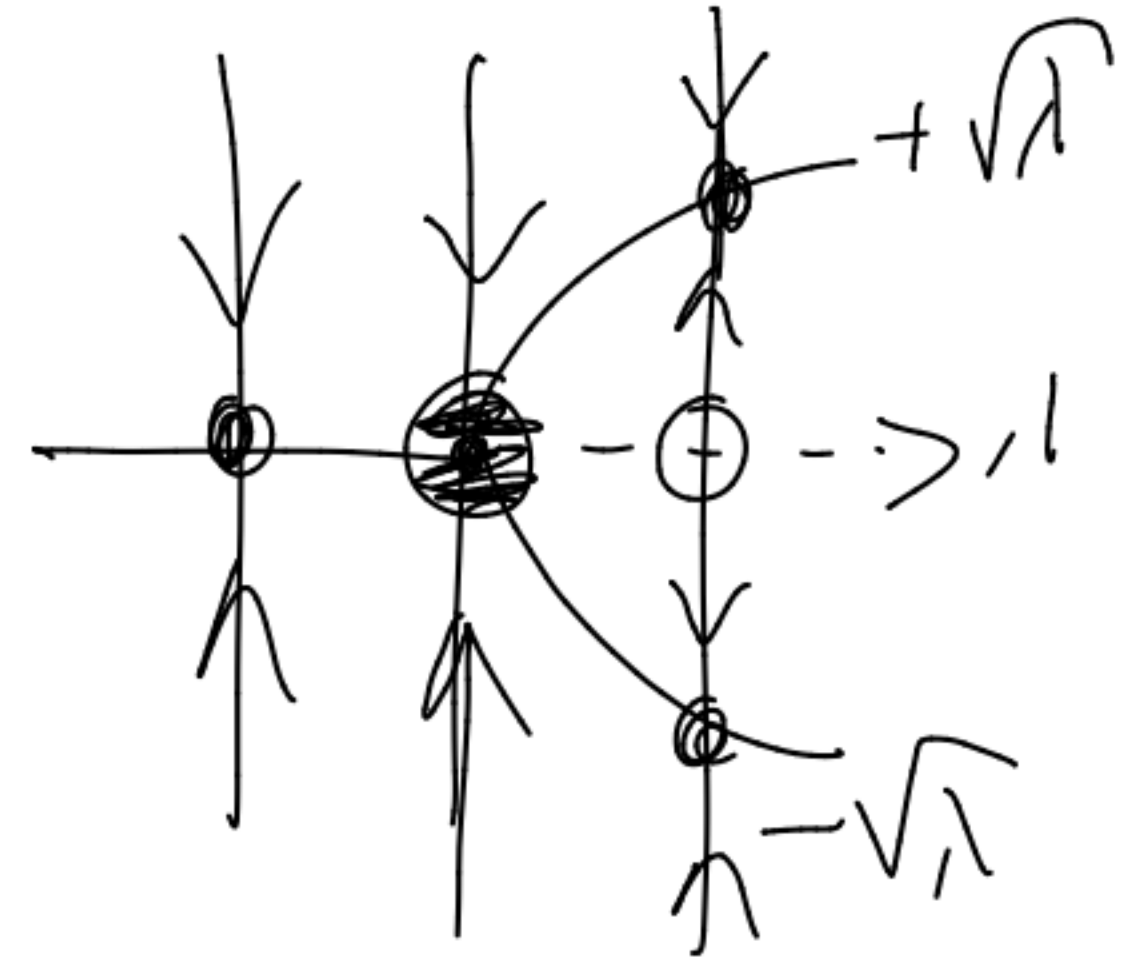
S2 Stability

$$f'(x) = \lambda - 3x^2$$

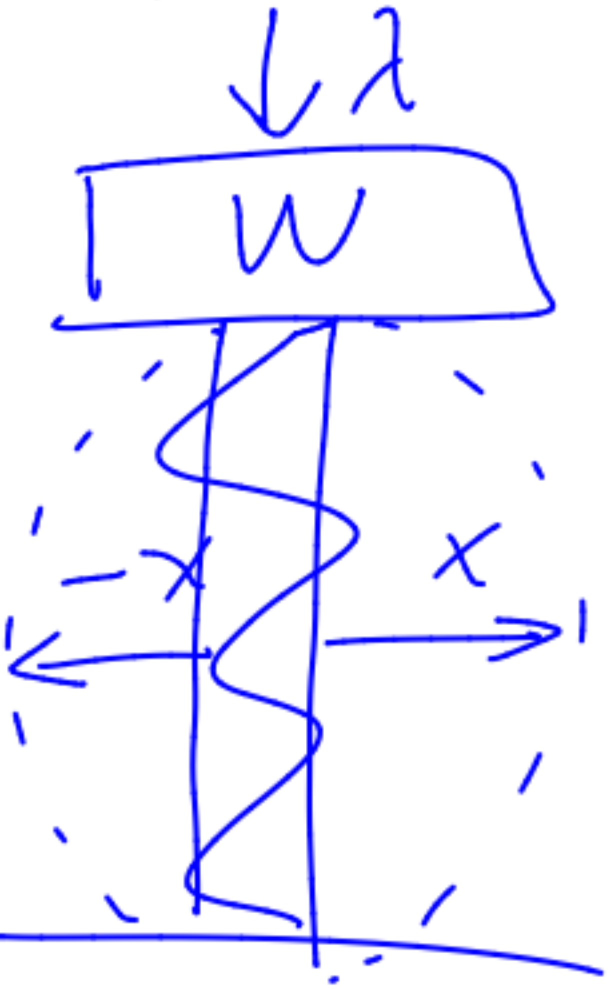
$$f'(x_1=0) = \lambda \Longrightarrow x_1 = 0 \begin{cases} \text{stable}, & \lambda < 0 \\ \text{unstable}, & \lambda > 0 \end{cases}$$

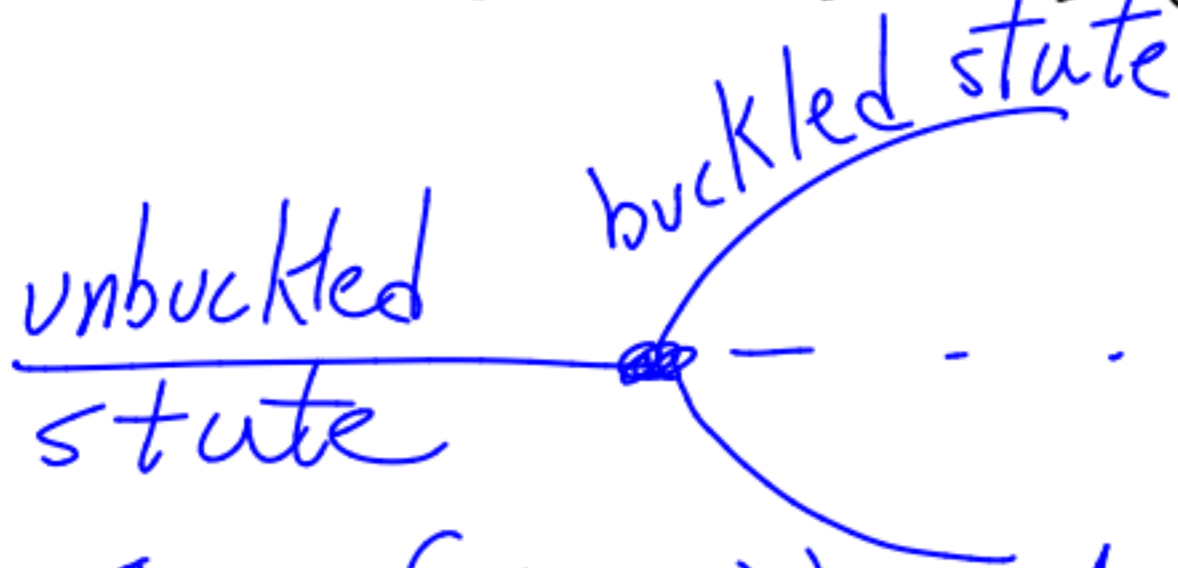
$$f'(x_{2,3} = \pm\sqrt{\lambda}) = -2\lambda \Rightarrow x_{2,3} = \pm\sqrt{\lambda} \begin{cases} \text{stable}, & \lambda > 0 \\ \underline{\qquad\qquad} \end{cases}$$

S3 Bif. Diag



Application: Euler's Beam Column





$\dot{x} = f(x,\lambda)$,

Assume: $f(x,\lambda) = 0$

By symmetry: $f(-x,\lambda) = 0$

Suppose: $\boxed{f(-x,\lambda) = -f(x,\lambda)} \Rightarrow f$ is odd

then $f(0,\lambda) = -f(0,\lambda) \Rightarrow f(0,\lambda) = 0$

$\Rightarrow x = 0$ is always an equil.

Taylors Thm.

$\Rightarrow \boxed{f(x,\lambda) = b(x,\lambda)\,x}$

$\Rightarrow f(-x,\lambda) = b(-x,\lambda)(-x) = -b(-x,\lambda)\cancel{x} \overset{\text{want}}{=} \overbrace{-b(x,\lambda)\cancel{x}}^{-f(x,\lambda)}$

$\Rightarrow b(-x,\lambda) = b(x,\lambda) \Rightarrow b$ is even $\Rightarrow \boxed{b(x,\lambda) = a(x^2,\lambda)}$

$$f(x,\lambda) = a(x^2,\lambda)\, x = \left[a(0,0) + \frac{\partial a}{\partial x}\Big|_{(0,0)} x^2 + \dots\right] + x \dots$$

$$\textcircled{0}\ \cancel{f}(0,0) = 0 \checkmark$$

$$f_x(0,0) = 0 \Leftrightarrow a(0,0) = 0$$

$$f(x,\lambda) \overset{\text{Taylor}}{=} \left[\alpha x^2 + \beta\lambda + \dots \text{h.o.t.}\right] x$$

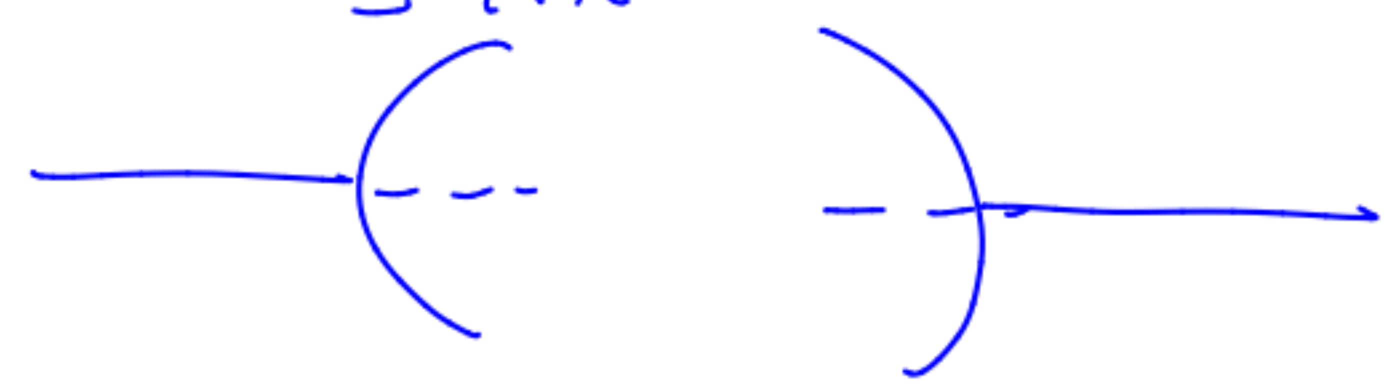
$$\alpha \neq 0 \qquad \beta \neq 0$$

$$\frac{dx}{dt} = \alpha x^3 + \beta\lambda x, \quad \text{Let } \tau = \alpha t \Rightarrow \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \alpha\frac{dx}{d\tau}$$

$$\cancel{\alpha}\frac{dx}{d\tau} = \cancel{\alpha} x^3 + \beta\lambda x \Rightarrow \frac{dx}{d\tau} = x^3 + \left(\frac{\beta\lambda}{\alpha}\right) x = x^3 \pm \tilde{\lambda} x$$

$$\pm \lambda x$$

$$\boxed{\frac{dx}{d\tau} = (x^2 \pm \tilde{\lambda})\, x}$$



Codim-one Steady State Bif.

$\dot{x} = f(x, \lambda)$

Bif. Cond: $f(0,0) = 0$

$f_x(0,0) = 0$

Bifurcation	Nondegeneracy Cond	Normal Form
Saddle-node	$f_\lambda(0,0) \neq 0$ $f_{xx}(0,0) \neq 0$	$\dot{x} = \lambda \pm x^2$
Transcritical	$f_\lambda(0,0) = 0$ $f_{x\lambda}(0,0) \neq 0$, $f_{xx}(0,0) \neq 0$	$\dot{x} = \lambda x \pm x^2$
Pitchfork	$f_\lambda(0,0) = 0$ $f_{xx}(0,0) = 0$, $f_{x\lambda}(0,0) \neq 0$ $f_{xxx}(0,0) \neq 0$	$\dot{x} = (x^2 \pm \lambda) x$

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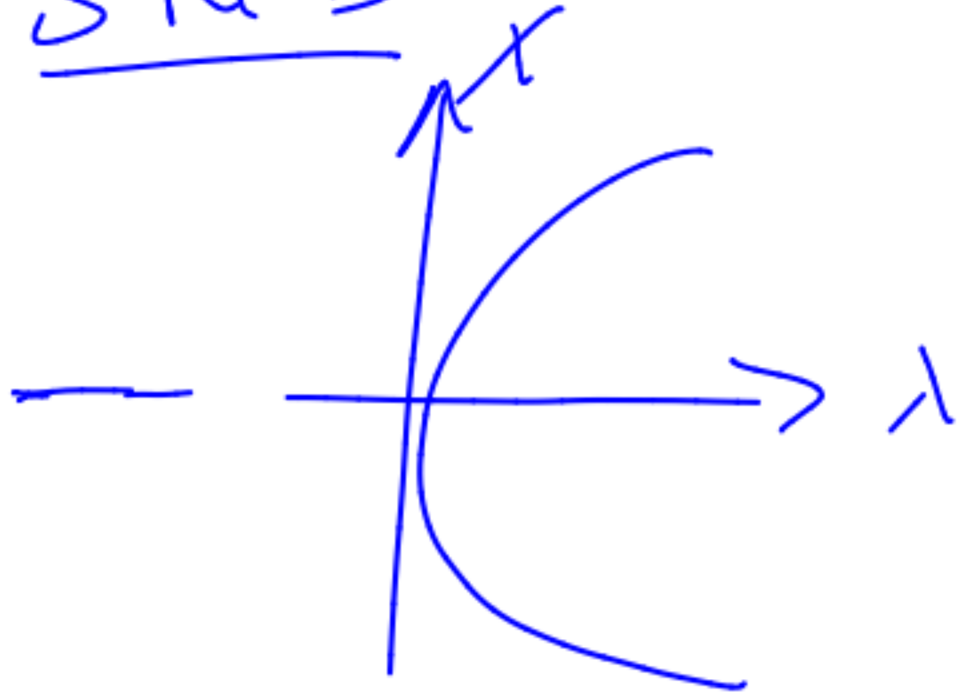
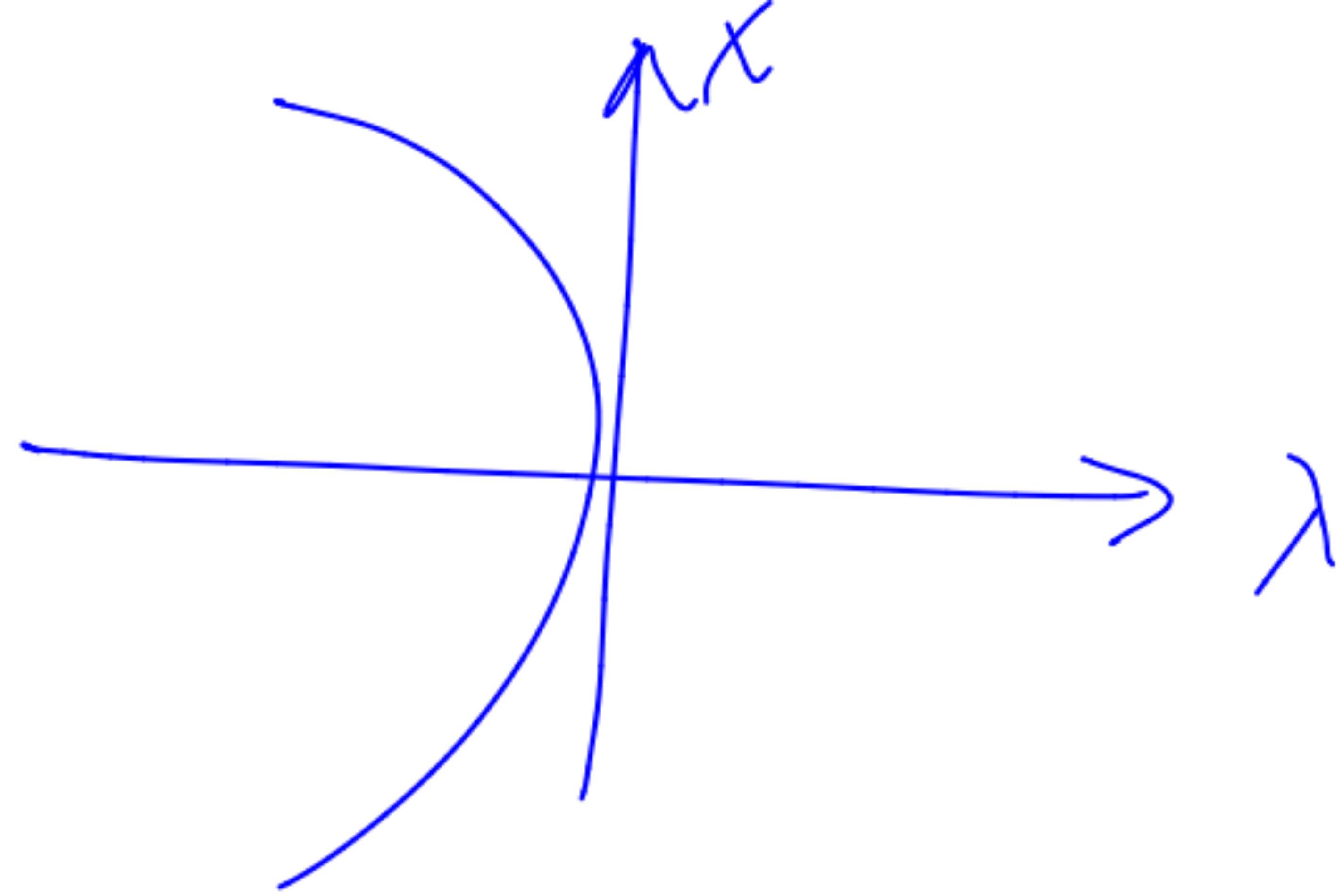
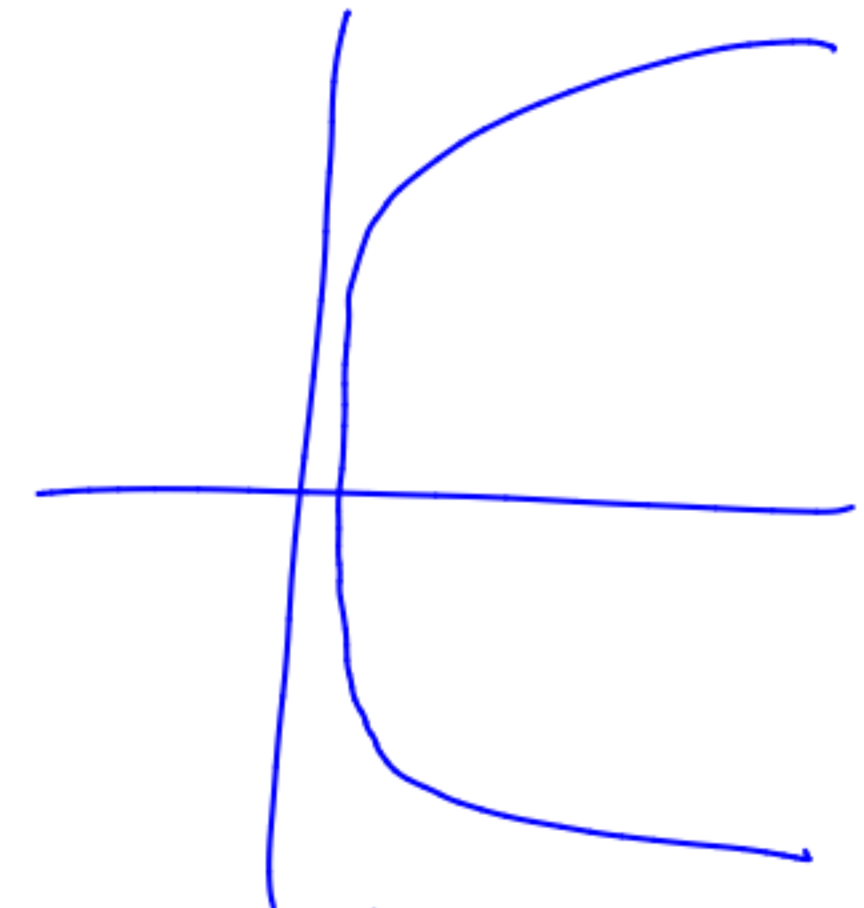


$$-\frac{f_{xx}(0,0)}{f_\lambda(0,0)} > 0$$



$$-\frac{f_{xx}(0,0)}{f_\lambda(0,0)} < 0$$



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