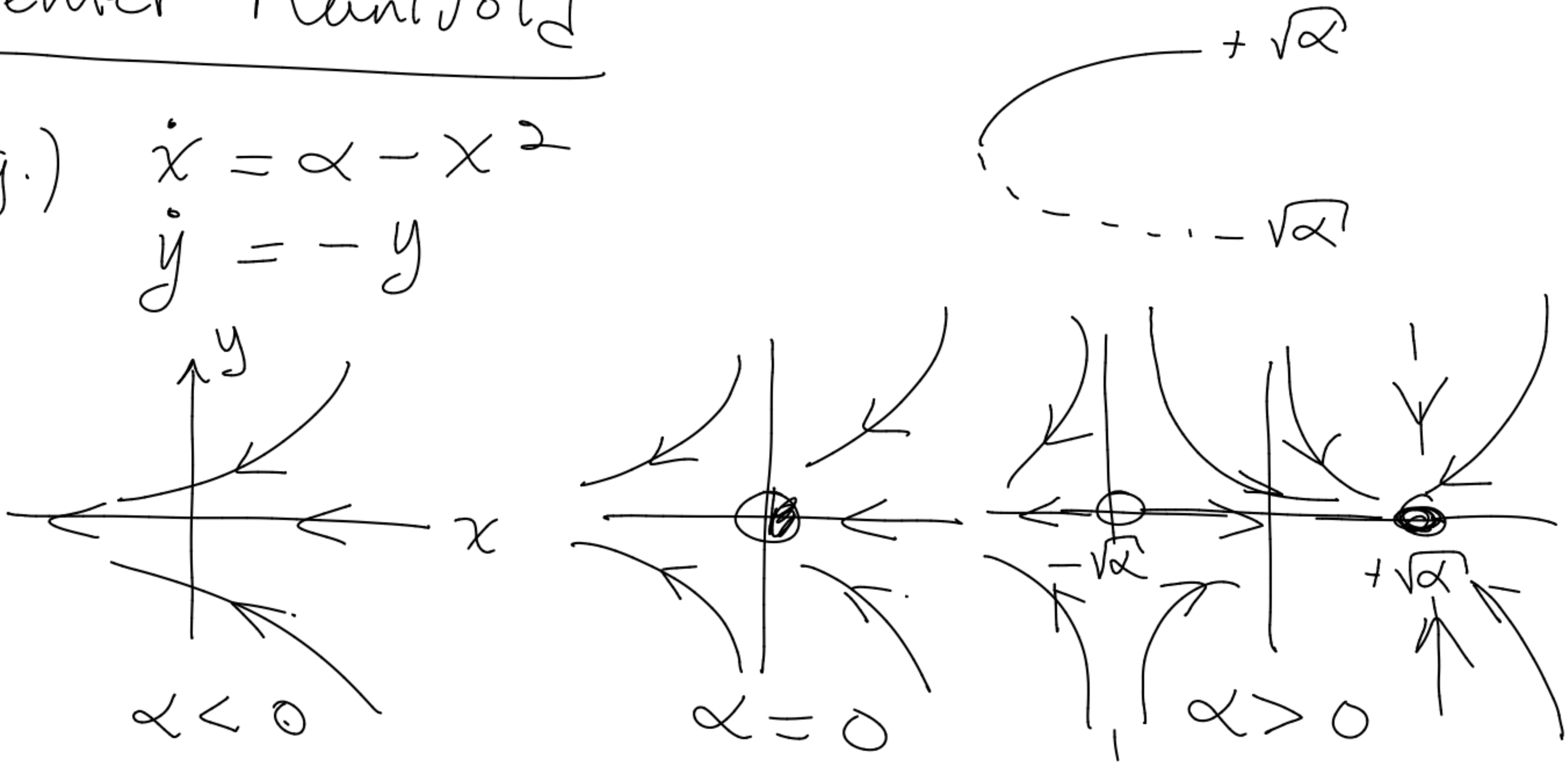
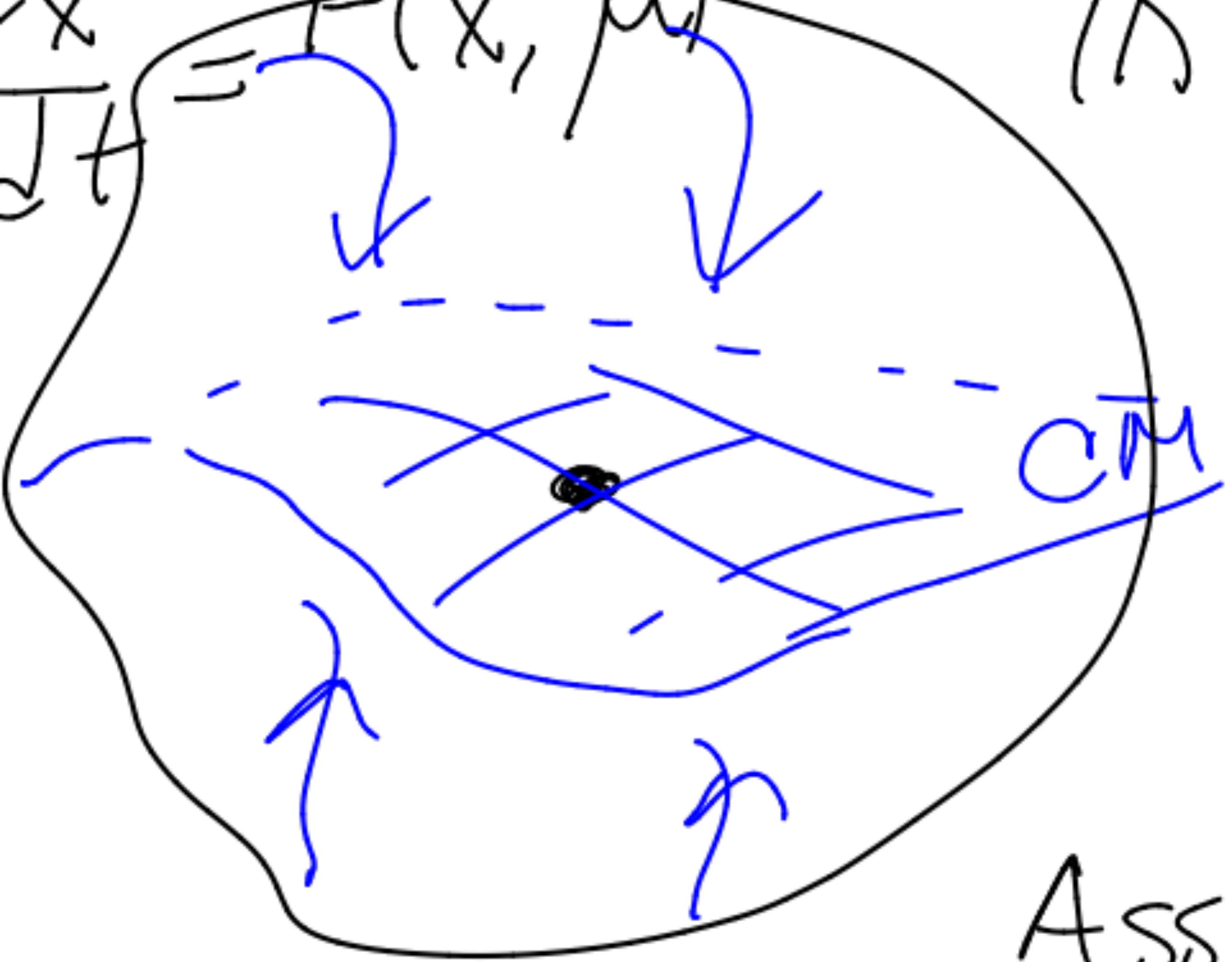
Center Manifold

e.g.) $\dot{x} = \alpha - x^2$
$\dot{y} = -y$



$$\frac{dx}{dt} = F(x, \mu) \qquad \mathbb{R}^n$$



Let

$$\dot{x} = Ay + f(x,y)$$
$$\dot{y} = By + g(x,y), \qquad (x,y) \in \mathbb{R}^c \times \mathbb{R}^s$$

Assume: $f(0,0) = g(0,0) = 0$

$Df(0,0) = Dg(0,0) = 0$

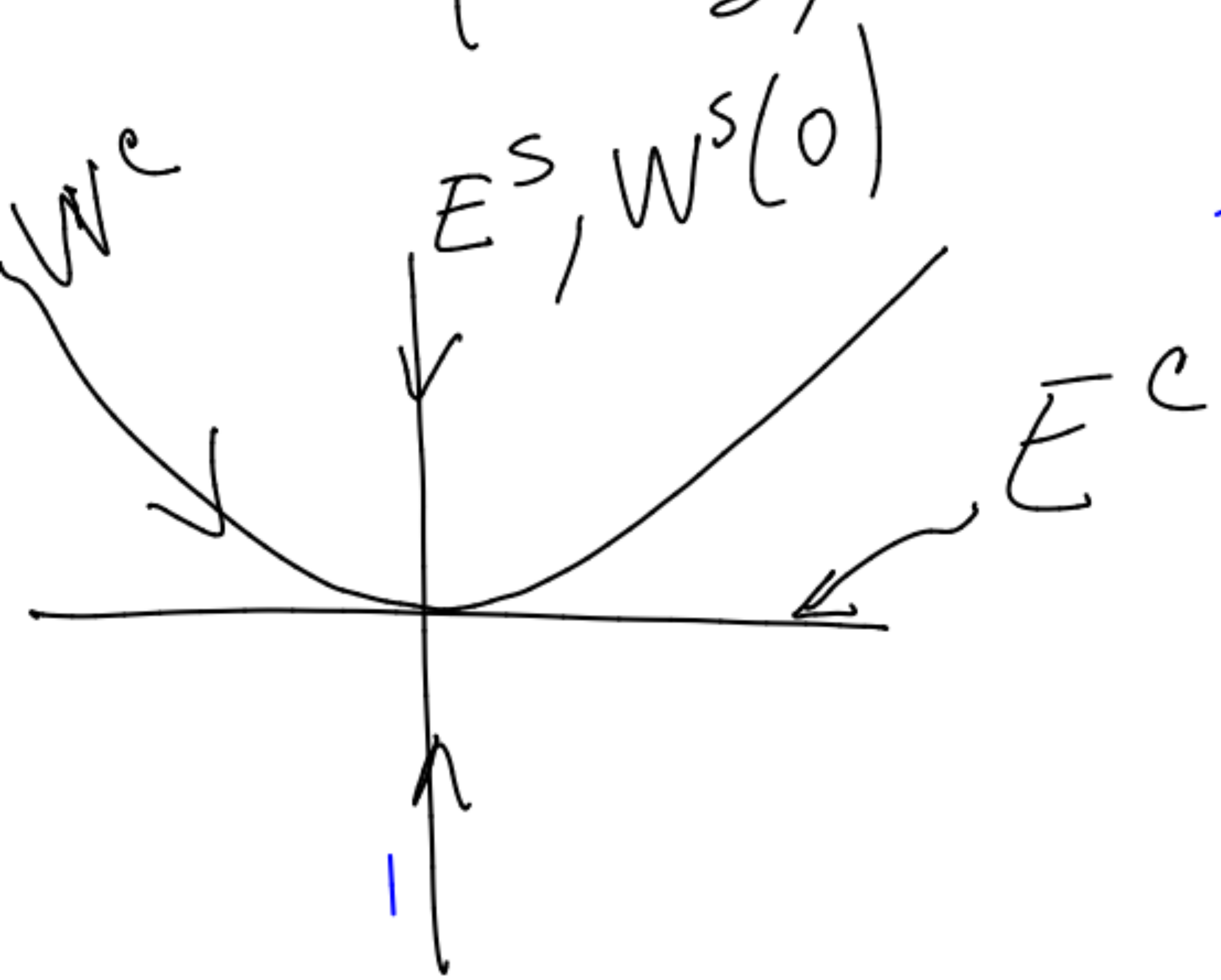
$A_{c \times c}$ — e-vals with zero real part

$B_{s \times s}$ — " negative "

Def An invariant manifold is a Center Manifold for (1) if it can be locally represented as

$$W^c = \{ (x,y) \in \mathbb{R}^c \times \mathbb{R}^s : y = h(x), \ |x| < \delta, \ h(0)=0, \ Dh(0)=0 \}$$

($h(0)=0$, $Dh(0)=0$: W^c is tangent to E^c)



Thm There exists a C^r C.M. for (1) where the dynamics of (1), restricted to CM, is

$$\dot{u} = Au + f(u, h(u)) \qquad (2)$$

heorem

(i) Suppose Equil. of (2) is stable (unstable)

Then the " (1) "

(i) Suppose Equil. of (2) is stable

Then if $(x(t), y(t))$ is a sol. of (1) with $(x(0), y(0))$ small, then there is a sol. $u(t)$ of (2) st. as $t \to \infty$:

$$x(t) = u(t) + O(e^{-\gamma t})$$
$$y(t) = h(u(t)) + O(e^{-\gamma t})$$

$\lim_{t\to\infty} y(t) = h(x)$

where γ is a constant

Algorithm

S1 We seek: $y = h(x)$

S2 Diff. w.r.t. t: $\dot{y} = Dh(x)\,\dot{x}$ (3)

S3 Substitute $y = h(x)$ in (1)

$$\dot{x} = Ax + f(x, h(x))$$

$$\dot{y} = Bh(x) + g(x, h(x))$$

Substituting into (3):

$$Dh(x)\left[Ax + f(x, h(x))\right] = Bh(x) + g(x, h(x))$$

$$N(h(x)) = Dh(x)\left[Ax + f(x,h)\right] - Bh - g(x,h) = 0$$

e.g.) $\dot{x} = x^2 y - x^5$

$\dot{y} = -y + x^2$

S1 $E_1(0,0)$

S2 $J = \begin{bmatrix} 2xy - 5x^4 & x^2 \\ 2x & -1 \end{bmatrix}$

$J|_{E_1} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ (A)

S3 $\dot{x} = 0\,x + (x^2 y - x^5)$ (f)

$\dot{y} = -1\cdot y + (x^2)$ (g) (B)

Let $y = h(x) = ax^2 + bx^3 + \dots$

$N(h) = (2ax + 3bx^2 + \dots)\left[0 + ax^4 + bx^5 - x^5\right] + (ax^2 + bx^3 + \dots) - x^2 \overset{\text{set}}{=} 0$

Equating like powers of x:

x^2: $a - 1 = 0$

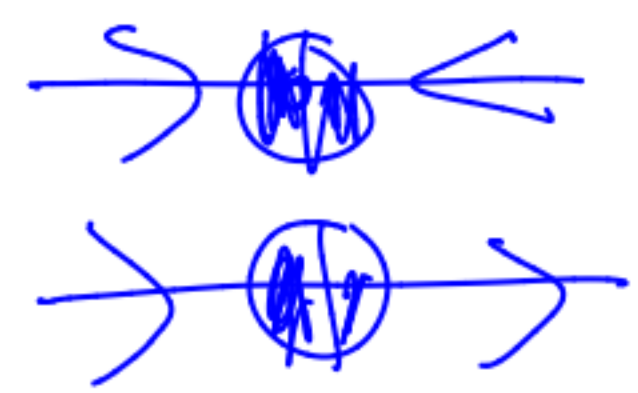
x^3: $b = 0$

$h(x) = x^2 + O(x^4)$

Thm $\Rightarrow$ $\dot{u} = 0 \cdot u + u^4 + O(x^5)$

⓪ $\dot{x} \approx -x^5$

$\dot{u} = u^4$



e.g.) Lorenz Model

bif par

$$\dot{x} = \sigma(y-x)$$
$$\dot{y} = \bar{\rho}x + x - y - xz$$
$$\dot{z} = -\beta z + xy$$

s1 $E_1(0,0,0)$

s2 $J\big|_{E_1} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}$,

e-vals: $0, \ -\sigma-1, \ -\beta$

e-vec: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Let $P = \begin{bmatrix} 1 & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -(\sigma+1) & 0 \\ 0 & 0 & -\beta \end{bmatrix}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \frac{1}{\sigma+1} \begin{bmatrix} \sigma\bar{\rho}(u+\sigma v) - \sigma w(u+\sigma v) \\ -\bar{\rho}(u+\sigma v) + w(u+\sigma v) \\ (1+\sigma)(u+\sigma v)(u-v) \end{bmatrix}$$

(B marks the lower-right block; f marks the first row, g the last two rows)

Let $\dot{\bar{\rho}} = 0$

$$v = h_1(u,\bar{\rho}) = a_1u^2 + a_2u\bar{\rho} + a_3\bar{\rho}^2 + \cdots$$
$$w = h_2(u,\bar{\rho}) = b_1u^2 + b_2u\bar{\rho} + b_3\bar{\rho}^2 + \cdots$$

CM Algorithm

$$h_1(u,\bar{e}) = -\frac{1}{(1+\sigma)^2}u\bar{e} + \ldots,$$

$$h_2(u,\bar{e}) = \frac{1}{\beta}u^2 + \ldots,$$

$$\dot{u} = \frac{1}{1+\sigma}u\left(\sigma\bar{e} - \frac{\sigma}{\beta}u^2 + \ldots\right)$$

$$\dot{\bar{e}} = 0$$