Abstract Groups

Def. A group Γ is a set $\{\gamma_1, \gamma_2, \dots\}$ that satisfies:

(i) Closure: $\Gamma \times \Gamma \to \Gamma$ or $\gamma_1 \cdot \gamma_2 = \gamma_3 \in \Gamma$

(ii) Associativity: $(\gamma_1 \gamma_2)\gamma_3 = \gamma_1(\gamma_2 \gamma_3)$

(iii) Identity: $\exists e \in \Gamma$ s.t. $\gamma e = e \gamma = \gamma, \ \forall \gamma \in \Gamma$

(iv) Inverses: $\exists \gamma^{-1} \in \Gamma \ \forall \gamma \in \Gamma$ s.t. $\gamma \gamma^{-1} = \gamma^{-1} \gamma = e$

Examples

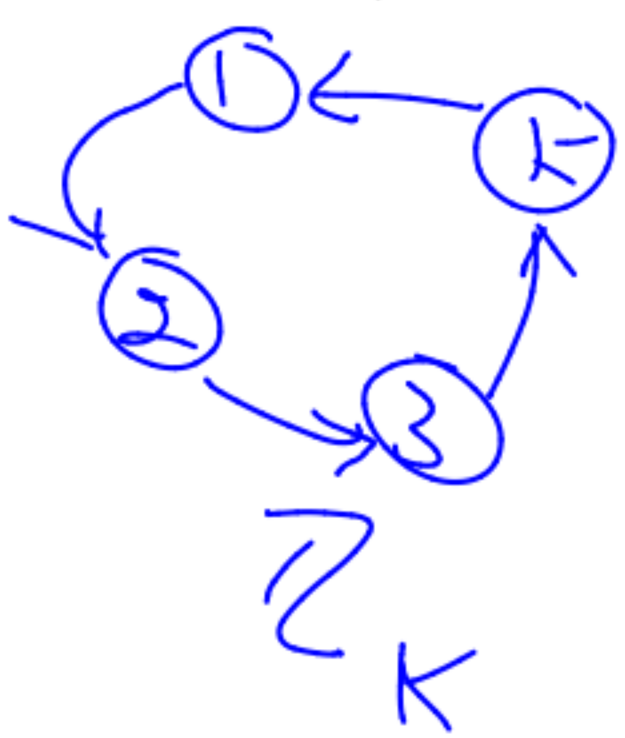
(a) $\mathbb{Z}_k = \{0, 1, \ldots, k-1\}$, "$\bullet$" $\equiv$ addition mod k

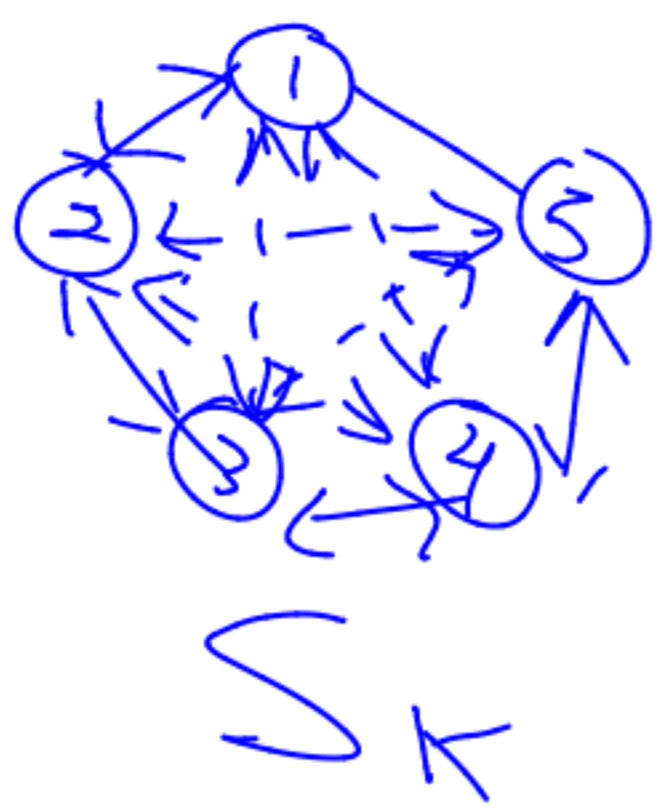
(b) S^1 = Circle group, "$\cdot$" $\equiv$ addition of angles

$= \{ z \in \mathbb{C} : |z| = 1 \}$

$\theta \in S^1$ or $e^{\theta i} \in S^1$, add modulo 2π: $e^{\theta i} e^{\varphi i} = e^{(\theta+\varphi)i}$

(c) $S_k \equiv$ Permutation on k letters/objects





(d) $GL(n) \equiv$ General Linear group

$\equiv$ Set of all real invertible $n \times n$ matrices

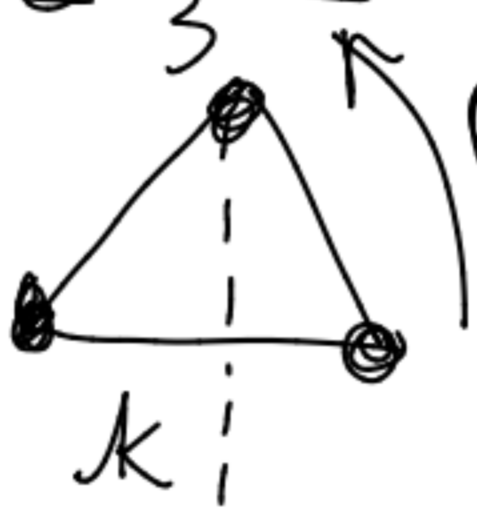
" $\bullet$ " $\equiv$ matrix multiplication

(e) $O(n) \equiv$ Set of $n \times n$ orthogonal matrices

$$\equiv \{ A \in \mathbb{R}^{n \times n} : A^T = A^{-1} \text{ or } AA^T = I_n \}$$

Def The order of a finite group is the number of elements that it contains

(f) $D_3 \equiv$ Dihedral group of a triangle.



$\rho = \frac{2\pi}{3}$ $\quad D_3 = \{ e, \rho, \rho^2, k, k\rho, k\rho^2 \} = \{ \{ \rho, k \} \}$ generators

Group Table (Cayley Table)

	e	r_1	$\cdots$	r_n
e	e	r_1		r_n
r_1	r_1	r_1^2	$r_1 r_2 \cdots$	$r_1 r_n$
$\vdots$	$\vdots$		$\ddots$	$\vdots$
r_n	r_n	$r_n r_1$	$\cdots$	r_n^2

e.g.) $\mathbb{Z}_2 = \{\overset{I}{0}, \overset{\kappa}{1}\}$

	0	1
0	0	1
1	1	0

e.g.) $\mathbb{Z}_2 = \{-1, 1\}$

	1	-1
1	1	-1
-1	-1	1

	I	κ
I	I	κ
κ	κ	I

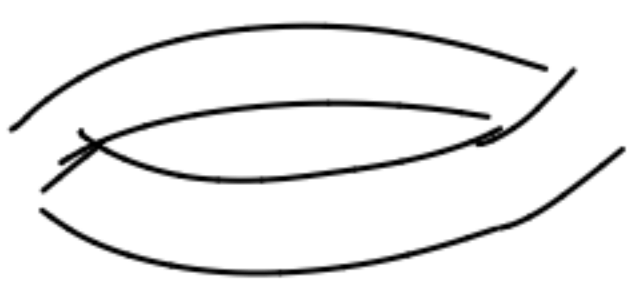
$2 : \quad x \longmapsto -x$

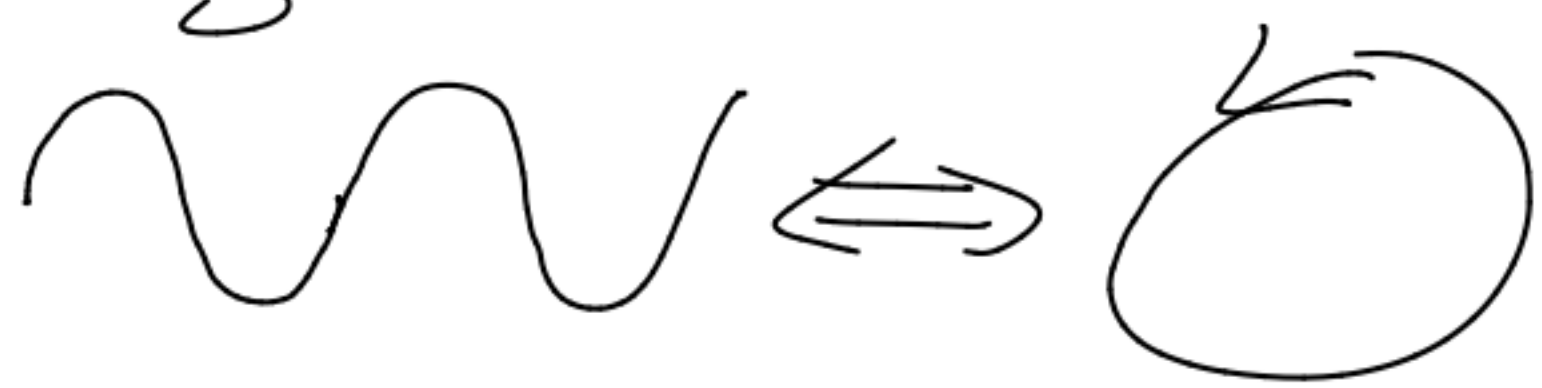
$\mathbb{Z}_2 = \{I, \kappa\}$, $\kappa(x,y) = (x,-y)$

$SO(2) \equiv$ Special orthogonal group in $\mathbb{R}^2$

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

⊙ $SO(2) \overset{\text{isomorphic}}{\sim} S^1$





$$T^m = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{m \text{ times}}$$

$E(2) \equiv$ Euclidean group in the plane (rotations, reflections, translations)

$SE(2) \equiv$ Special Euclidean Group (rotations, translations)

Finite Groups	Compact Continuous Groups	Noncompact Groups
— D_n	— S^1	— $E(2)$
— $\mathbb{Z}_n$	— $SO(2) \sim S^1$	— $SE(2)$
— S_n	— $O(n)$	
	— $SO(n)$	
	— $T^m = S^1 \times \dots \times S^1$	

Def A group Γ is Abelian if $\gamma_1\gamma_2 = \gamma_2\gamma_1$, $\forall \gamma_1, \gamma_2 \in \Gamma$

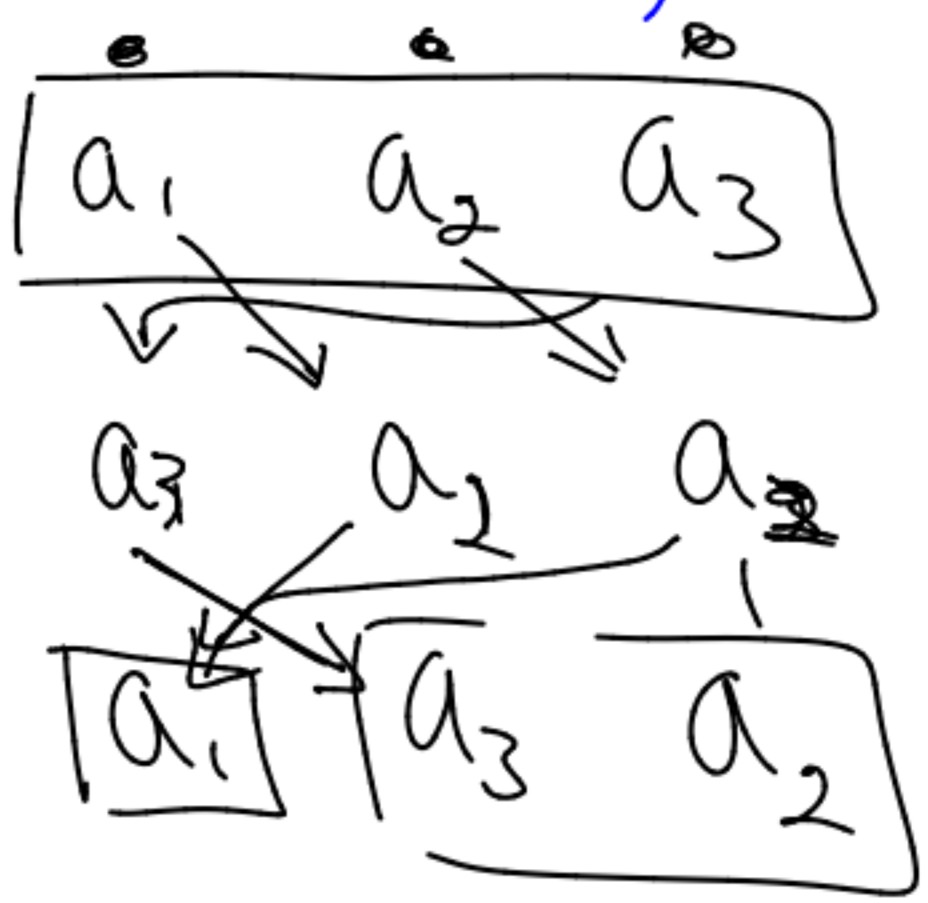
e.g.) $\mathbb{Z}_K$, S^1 are Abelian

S_K $(K \geq 3)$ is not Abelian

$$(1\ 2)(1\ 2\ 3) = (1)(1\ 2\ 3) = (2\ 3)$$

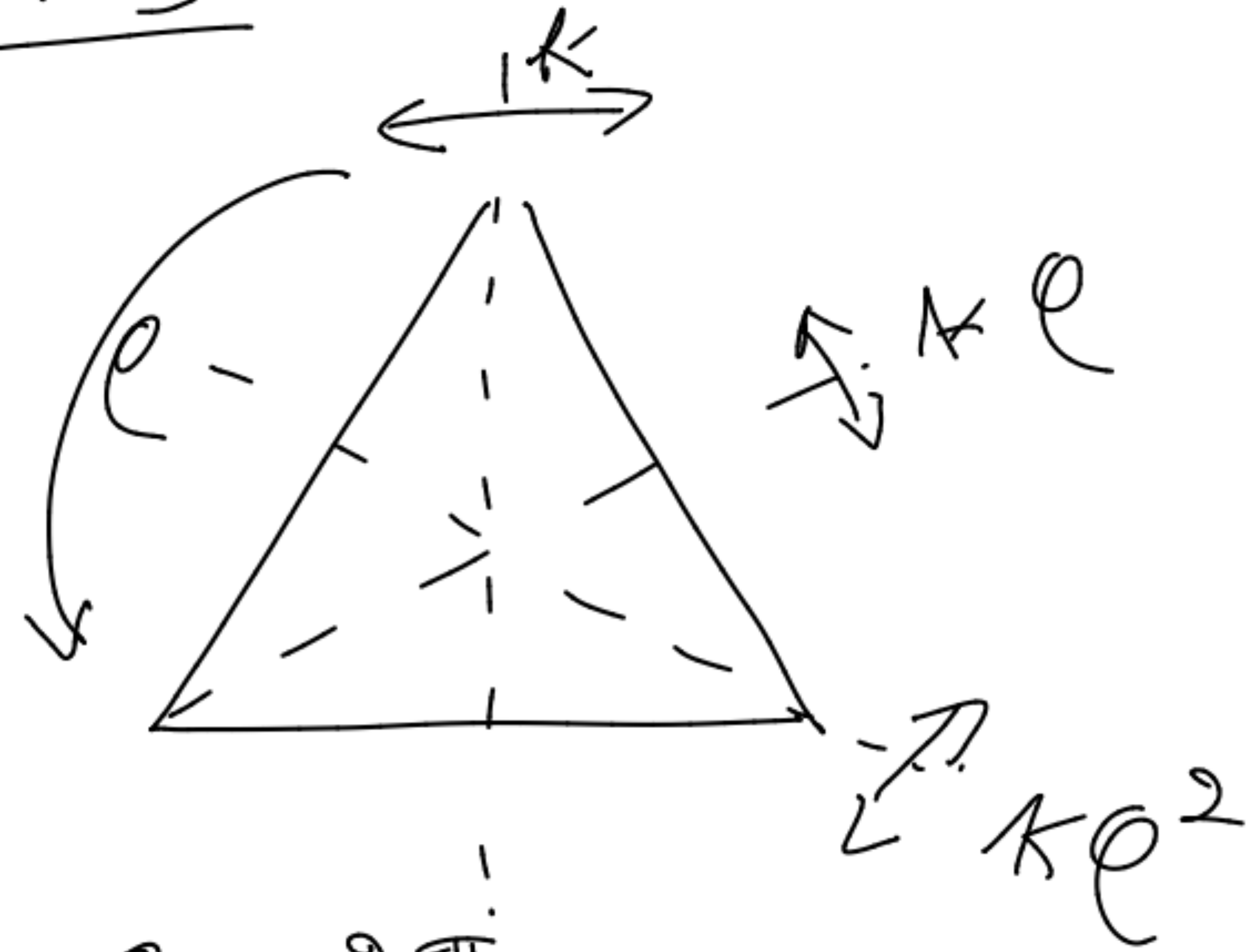
$$(1\ 2\ 3)(1\ 2) = (1\ 3)(2) = (1\ 3)$$

$(1\ 2\ 3)$



$(1\,2)(1\,2\,3)$

D_3



$\rho \equiv \frac{2\pi}{3}$

	e	ρ	ρ^2	κ	$\kappa\rho$	$\kappa\rho^2$
e	e	ρ	ρ^2	κ	$\kappa\rho$	$\kappa\rho^2$
ρ	ρ	ρ^2	e	$\kappa\rho^2$	κ	$\kappa\rho$
ρ^2	ρ^2	e	ρ	$\kappa\rho$	$\kappa\rho^2$	κ
κ	κ	$\kappa\rho$	$\kappa\rho^2$	e	ρ	ρ^2
$\kappa\rho$	$\kappa\rho$	$\kappa\rho^2$	κ	ρ^2	e	ρ
$\kappa\rho^2$	$\kappa\rho^2$	κ	$\kappa\rho$	ρ	ρ^2	e

Def Order n of an element $\gamma \in \Gamma$ is the smallest integer n s.t. $\gamma^n = e$,

$\rho^2 \cdot \kappa\rho \neq \kappa\rho \cdot \rho^2$

◎ D_3 is not Abelian

E(2)

Let $(\rho, t) \equiv \left(\begin{array}{l} \text{Reflection in an axis containing } o \\ \text{OR} \\ \text{Rotation about origin} \end{array}, \text{translation} \right)$

$$x' = (\rho, t)x = \rho x + t$$

$$x'' = (\rho_1, t_1)(\rho_2, t_2)x = (\rho_1, t_1)(\rho_2 x + t_2)$$

$$= \rho_1(\rho_2 x + t_2) + t_1 = \rho_1 \rho_2 x + \rho_1 t_2 + t_1$$

$$(\rho_1\rho_2, t_1) = (\rho_1\rho_2, \rho_1 t_2 + t_1) = \rho_1\rho_2 x + \rho_1 t_2 + t_1$$

$\Longrightarrow$ E(2) is closed

Subgroups

<u>Def</u> A subgroup $H \subseteq \Gamma$ is a <u>subset</u> that forms a group under the same group operation.

◎ Show H is closed, $e \in H$, $\exists \gamma^{-1} \in H$

e.g.) $\mathbb{Z}_3 \subset D_3$

$\mathbb{Z}_2 \subset D_3$