Def. H is a normal subgroup of Γ if

$$\gamma h \gamma^{-1} \in H, \quad \forall \gamma \in \Gamma, \ h \in H$$

$$H \lhd \Gamma$$

◎ If Γ is Abelian then all subgroups are normal since $\gamma h \gamma^{-1} = \gamma \gamma^{-1} h = h$

Def Complement $\Gamma / H \equiv \Gamma - H = \{\gamma \in \Gamma : \gamma \in H\}$

Def Normalizer $\equiv N(H) = \{\gamma \in \Gamma : \gamma^{-1} H \gamma = H\}$

e.g.) Let $A_n = \{\alpha \in S_n \mid \alpha \text{ is even}\}$

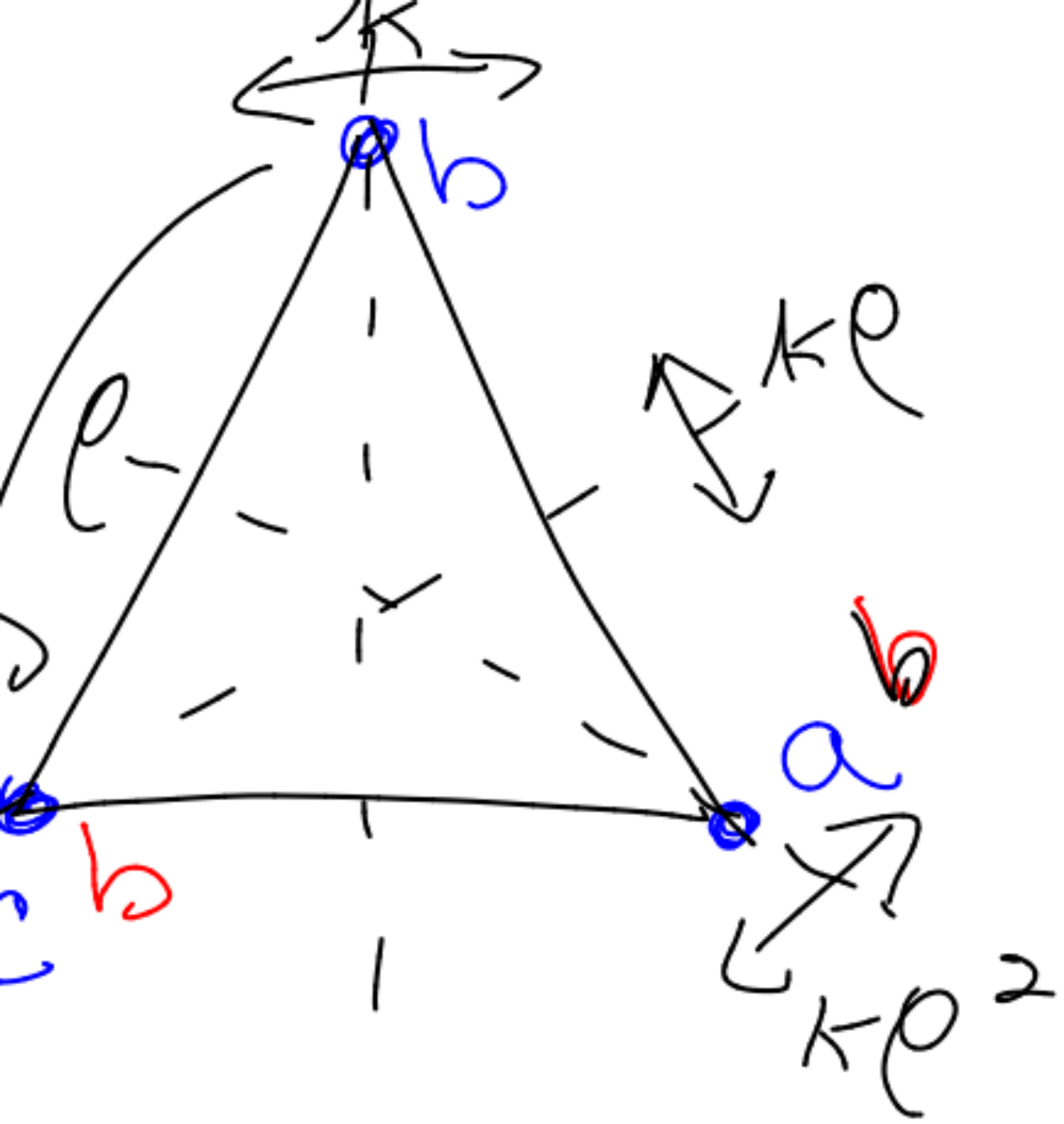
$\equiv$ Alternating group of order n

⑥ A permutation that can be expressed as a product of an even number of 2-cycles is even

$A_n \lhd S_n$

$$\underbrace{SL(2, \mathbb{R})}_{\substack{2\times 2 \text{ matrices} \\ \text{with } \det = 1}} \lhd \underbrace{GL(2, \mathbb{R})}_{\substack{2\times 2 \text{ matrices} \\ \text{with } \det \neq 0}}$$

⑤ $N(H) \equiv$ Largest subgroup of Γ that has H as a normal subgroup.



$^{2} \cdot \underbrace{k\rho \cdot b}_{a}$

	e	ρ	ρ^2	k	$k\rho$	$k\rho^2$
e	e	ρ	ρ^2	k	$k\rho$	$k\rho^2$
ρ	ρ	ρ^2	e	$k\rho^2$	k	$k\rho$
ρ^2	ρ^2	e	ρ	$k\rho$	$k\rho^2$	k
k	k	$k\rho$	$k\rho^2$	e	ρ	ρ^2
$k\rho$	$k\rho$	$k\rho^2$	k	ρ^2	e	ρ
$k\rho^2$	$k\rho^2$	k	$k\rho$	ρ	ρ^2	e

e.g.) $\Gamma = D_3, \quad H = \mathbb{Z}_2(\kappa) = \{e, \kappa\}$

$$N(\mathbb{Z}_2(\kappa)) = \{\gamma \in D_3 : \gamma^{-1}\mathbb{Z}_2(\kappa)\gamma = \mathbb{Z}_2(\kappa)\}$$

$$\mathbb{Z}_2(\kappa)\gamma = \gamma\mathbb{Z}_2(\kappa)$$

$$= \mathbb{Z}_2(\kappa)$$

$$H = \mathbb{Z}_3 = \{e, \rho, \rho^2\} \subset D_3$$

$$N(\mathbb{Z}_3) = \{\gamma \in D_3 : \gamma^{-1}\mathbb{Z}_3\gamma = \mathbb{Z}_3\} = D_3$$

$$\mathbb{Z}_3\gamma = \gamma\mathbb{Z}_3$$

$$\mathbb{Z}_3 \cdot \kappa \overset{?}{=} \kappa\mathbb{Z}_3$$

$$\underset{\kappa\rho^2}{\underset{\parallel}{\rho\cdot\kappa}} \neq \underset{\kappa\rho}{\underset{\parallel}{\kappa\rho}}$$

Def. Let $H \leq \Gamma$

$$\text{Left coset} \equiv \gamma H = \{\gamma h \mid h \in H\}$$

$$\text{Right coset} \equiv H\gamma = \{h\gamma \mid h \in H\}$$

6 - Cosets are disjoint

- Cosets partition a group Γ.

e.g.) $\Gamma = D_3 = \{e, \rho, \rho^2, \kappa, \kappa\rho, \kappa\rho^2\}$

$H = Z_3 = \{e, \rho, \rho^2\}$

	H	κH
H	H	κH
κH	κH	H

Def Let $H \triangleleft G$

$$G/H = \{\underbrace{aH \mid a \in G}_{\text{set of disjoint cosets}}\}$$

$\odot$ $(\gamma_1 H)(\gamma_2 H) = \gamma_1\gamma_2 H, \ \forall \gamma_1, \gamma_2 \in \Gamma$

e.g.) $D_3/\mathbb{Z}_3 \simeq \mathbb{Z}_2(\mathcal{K})$

Def $h_1, h_2 \in \Gamma$ are conjugate if $\exists \gamma \in \Gamma$ s.t. $h_1 = \gamma h_2 \gamma^{-1}$

Def $H_1 \overset{\text{conjugate}}{\sim} H_2$ if $\exists \gamma \in \Gamma$ s.t. $H_1 = \gamma H_2 \gamma^{-1}$

e.g.) $\Gamma = D_3$

$$\rho \sim \rho^2 \quad \text{since} \quad \rho^2 = \kappa \rho \kappa^{-1} \text{ or } \rho^2 \kappa = \kappa \rho$$

$$\odot \quad \text{order}(\rho^2) = \text{order}(\rho) = 3$$

$$\left.\begin{array}{l} Z_2(\kappa) = \{e, \kappa\} \\ \tilde{Z}_2(\kappa) = \{e, \kappa\rho\} \end{array}\right\} \quad \begin{array}{l} Z_2(\kappa) \sim \tilde{Z}(\kappa) \\ \rho \kappa \rho^{-1} = \kappa \rho \\ \rho e \rho^{-1} = e \rho \end{array}$$

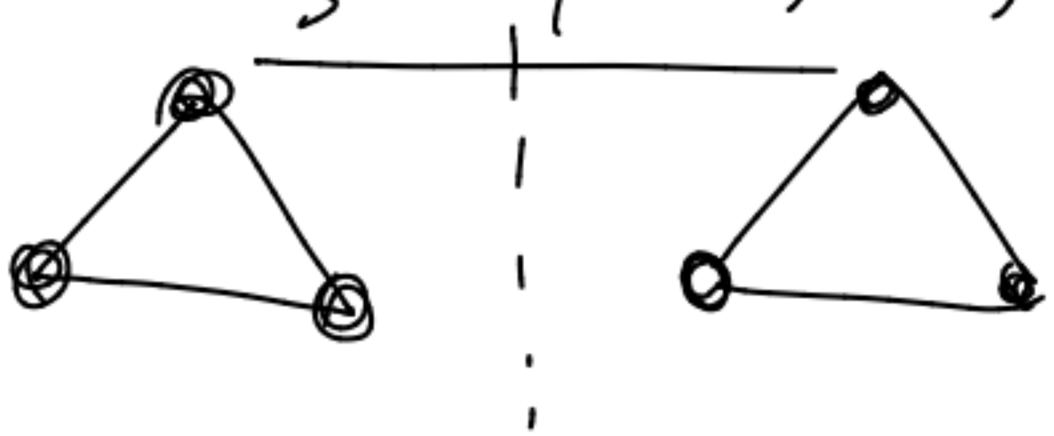
Products of Groups

Def $\Gamma \underbrace{\times}_{\text{external product}} \Delta \equiv \Gamma \underset{\text{direct sum}}{\oplus} \Delta = \{(\gamma, \delta) \mid \gamma \in \Gamma, \delta \in \Delta\}$

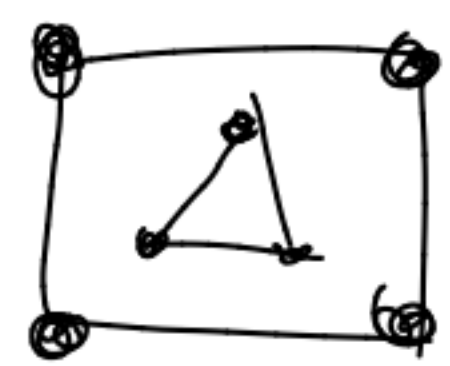
$$G_1 \oplus G_2 \oplus \cdots \oplus G_n = \{(g_1, \ldots, g_n) \mid g_i \in G_i\}$$

e.g.) $\mathbb{Z}_2 = \{0, 1\}$, $\mathbb{Z}_3 = \{0, 1, 2\}$

$$\mathbb{Z}_2 \oplus \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\} = \{(1,1)\}$$

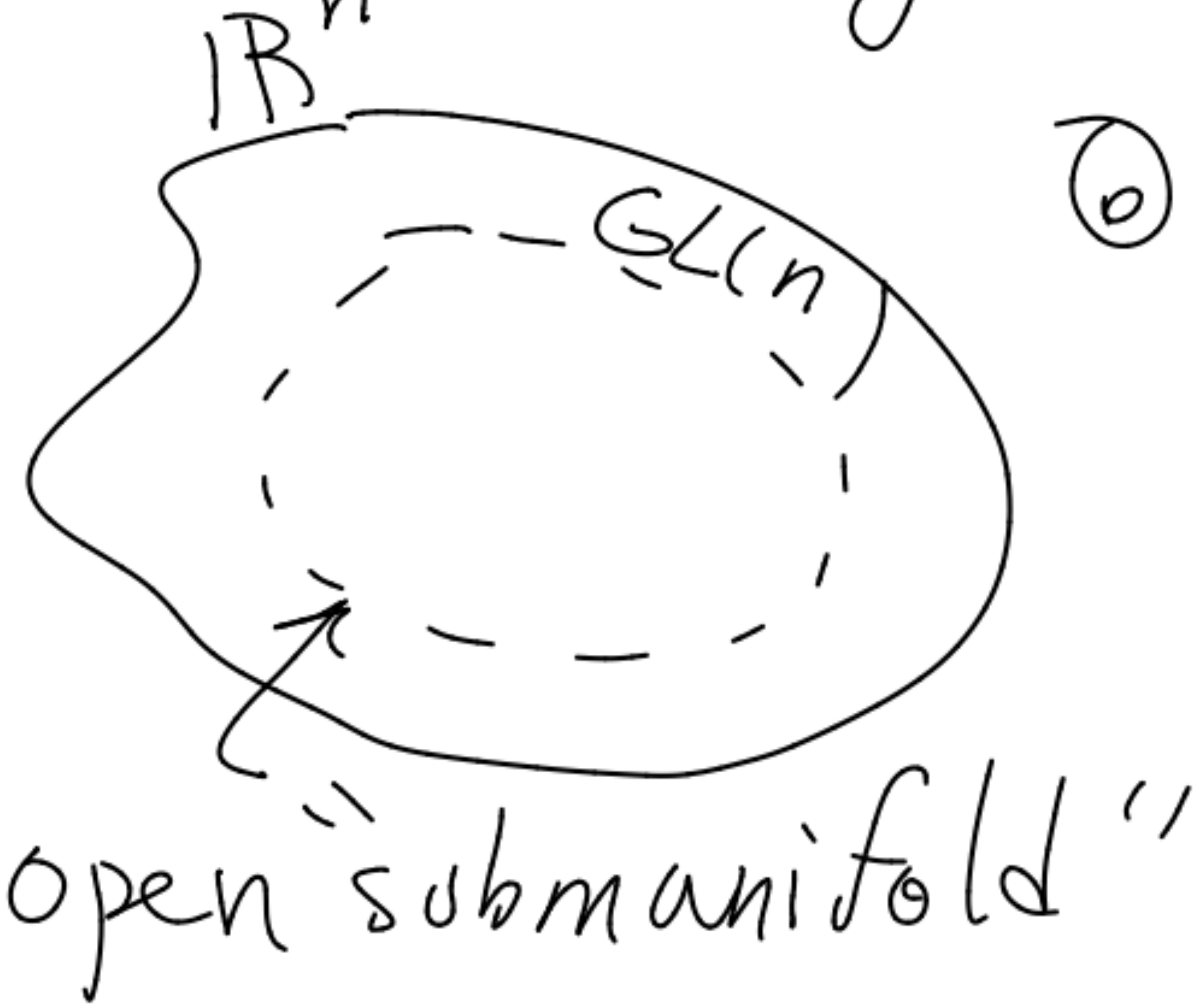


e.g.) $D_4 \oplus D_3$



Lie Groups

Def A Lie group is a closed subgroup of $GL(n)$



Γ is closed if it is a closed subset of $GL(n)$ as well as a subgroup of $GL(n)$.

Representation Theory (of groups)

Def (Action of a group)
Γ acts linearly on $V = \mathbb{B}^n$ if $\exists$ a continuous map $\Gamma \times V \longrightarrow V$

$$(\gamma, v) \rightarrow \gamma \cdot v = \rho_\gamma(v)$$

st.

1) ρ_γ is linear & invertible (matrix)
2) $\gamma_1(\gamma_2 v) = (\gamma_1 \gamma_2) v$
$\rho_{\gamma_1} \circ \rho_{\gamma_2} = \rho_{\gamma_1 \circ \gamma_2}$

ρ is a representation of Γ

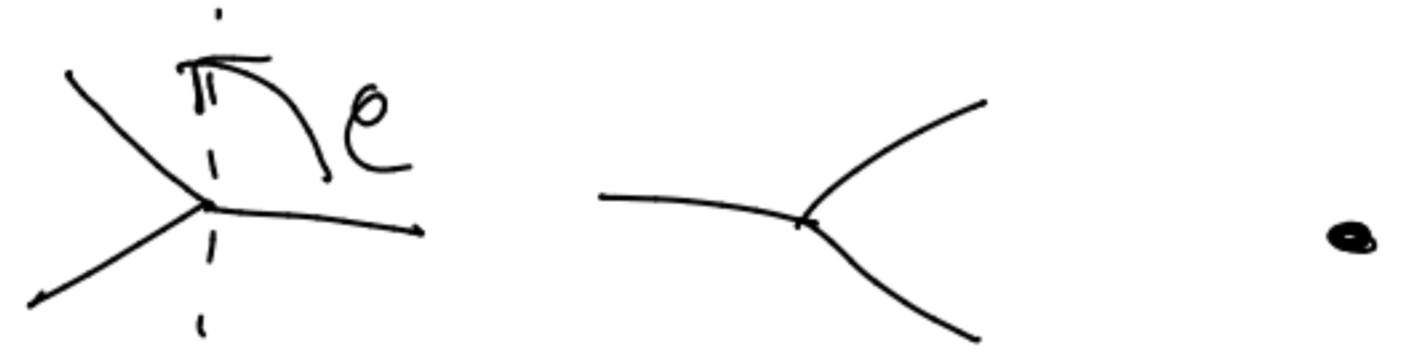
Def A representation of a finite group Γ over a field F is a homeomorphism

$$\theta : \Gamma \longrightarrow GL(n, F)$$

$\deg(\theta)$ or $\dim(\theta)$ is n

$\theta(\gamma) = M_\gamma$, $M_{\gamma_1} M_{\gamma_2} = M$

e.g.) $\mathbb{Z}_2 = \{0, 1\}$



$$\rho(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

e.g.) $D_3 = \{e, \rho, \rho^2, \kappa, \kappa\rho, \kappa\rho^2\}$

$$M_e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_\rho = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \quad M_{\rho^2} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$M_\kappa = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_{\kappa\rho} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \quad M_{\kappa\rho^2} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

Natural Rep. of D_3