e.g.) $\mathbb{Z}_2 = \{\overset{\nearrow I_2}{0}, \overset{\nearrow k}{1}\}$

	0	1
0	0	1
1	1	0

$\mathbb{Z}_2 = \{-1, 1\}$

	1	-1
1	1	-1
-1	-1	1

$x \mapsto -x$
$x \mapsto x$

$\mathbb{Z}_2 = \{I_2, k\}$
$k(x,y) = (x,-y)$

	I	k
I	I	k
k	k	I

$k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Representations of Groups

<u>Def.</u> Γ acts linearly on a vector space $V \equiv \mathbb{R}^n$ if $\exists$ a continuous map $\Gamma \times V \to V$

$(\gamma, v) \to \gamma \cdot v = \rho_\gamma(v)$ st.

(1) ρ_γ is linear & invertible (a matrix)

(2) $\gamma_1 \cdot (\gamma_2 v) = (\gamma_1 \gamma_2) v$ or $\rho_{\gamma_1} \circ \rho_{\gamma_2} = \rho_{\gamma_1 \gamma_2}$

ρ is a representation of Γ

<u>Def.</u> A representation of a finite group Γ over a field F is a homeomorphism $\theta : \Gamma \to GL(n, F)$

⊙ degree or dim. of θ is n ↳degree or dim. of θ

Loosely speaking, an n-dim representation of Γ is a set of invertible $n \times n$ matrices that conform to the group structure: $\theta(\gamma) = \underset{\text{matrix}}{\underline{M_\gamma}}$, $M_{\gamma_2} M_{\gamma_1} = M_{\gamma_2 \gamma_1}$, $\forall \gamma_1, \gamma_2 \in \Gamma$

<u>Examples.</u>

$\mathbb{Z}_2 = \{0, 1\}$, $\rho(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$D_3 = \{e, e, e^2, m, me, me^2\}$

$M_e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $M_m = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$M_e = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$, $M_{me} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

$M_{e^2} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$, $M_{me^2} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

Natural representation of D_3

Remark

Every group Γ has a one dim. identity or trivial rep. given by $M_{\gamma} = 1$, $\forall \gamma \in \Gamma$.

$\odot$ $M_{\gamma_1} M_{\gamma_2} = 1 \times 1 = 1 = M_{\gamma_1 \gamma_2}$, $\forall \gamma_1, \gamma_2 \in \Gamma$

e.g.) S^1 acting on $\mathbb{R}^2$

Fix integer k s.t. $\theta \cdot z = e^{ki\theta} z$

If $z = x + yi$ then $e^{ki\theta} z = [\cos(k\theta) + \sin(k\theta) i](x + yi)$

$= [\cos k\theta x - \sin(k\theta) y] + [\sin(k\theta) x + \cos(k\theta) y] i$

$$\Rightarrow \theta \cdot z = \underbrace{\begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix}}_{\rho_\theta \text{ or } M_\theta} \begin{bmatrix} x \\ y \end{bmatrix}$$

Def. A representation is faithful if $\theta : \Gamma \longrightarrow GL(n, F)$ is an isomorphism. e.g: natural rep. of D_3 is faithful but the id. rep. is not faithful.

Def. Two representations A_i & B_i are equivalent or isomorphic if $\exists$ invertible matrix S st.

$$A_i = S^{-1} B_i S$$

or $SA_i = B_i S$

$$\begin{tikzcd} V \arrow[r, "A_i"] \arrow[d, "S"'] \arrow[dr, bend left=60] \arrow[dr, bend right=60] & V \arrow[d, "S"] \\ W \arrow[r, "B_i"] & W \end{tikzcd}$$

$SA_i = B_i S$

Example

$\Gamma = D_3$,

$\rho_2 : z \longrightarrow e^{\frac{2\pi}{3} i} z$ or $\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} \cos \frac{2\pi}{3} & -\sin \frac{2\pi}{3} \\ \sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} \end{bmatrix}}_{M_{\rho_1}} \begin{bmatrix} x \\ y \end{bmatrix}$

Recall Standard or Natural Rep.

$m_1 : z \longrightarrow -\bar{z}$ or $\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} -x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{M_{m_1}} \begin{bmatrix} x \\ y \end{bmatrix}$

Let $\theta(z) = \theta(x + yi) = \begin{pmatrix} x \\ y \end{pmatrix}$, $\forall x, y \in \mathbb{R}$

homeomorphism

$$M_\rho\, \theta(x+yi) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{x}{2} - \frac{\sqrt{3}}{2} y \\ \frac{\sqrt{3}}{2} x - \frac{1}{2} y \end{bmatrix}$$

$$\theta\left(\rho_2(x+yi)\right) = \theta\left(e^{\frac{2\pi}{3}i}(x+yi)\right) = \theta\left(\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)(x+yi)\right)$$

$$= \theta\left(\left(-\frac{x}{2} - \frac{\sqrt{3}}{2}y\right) + \left(\frac{\sqrt{3}}{2}x - \frac{y}{2}\right)i\right)$$

$$= \begin{bmatrix} -\frac{x}{2} - \frac{\sqrt{3}}{2} y \\ \frac{\sqrt{3}}{2} x - \frac{1}{2} y \end{bmatrix}$$

$\Rightarrow M_{\rho_1}\theta = \theta \rho_2$

Likewise, $M_{m_1}\theta = \theta m_2$

Def. A subspace $W \subset V$ is Γ-invariant if

$$\underbrace{\underset{\text{matrix}}{\theta(\gamma)\, w \in W}}, \quad \forall \gamma \in \Gamma, \forall w \in W$$

or $\gamma W \subset W$

e.g.)
$W = \{0\}$ & $W = V$ are always Γ-invariant

Def. A rep. or action of Γ is irreducible if the only Γ-invariant subspaces are $\{0\}$ and V

⊚ Any 1D representation is irreducible

e.g.) Trivial rep. of Γ on $\mathbb{R}^1$: $\rho_\gamma(x) = x$

No proper subspaces of $\mathbb{R}^1 \Rightarrow$ Trivial rep. is irreducible

e.g.) Consider $\Gamma = S^1$ acting on $\mathbb{C} \cong \mathbb{R}^2$

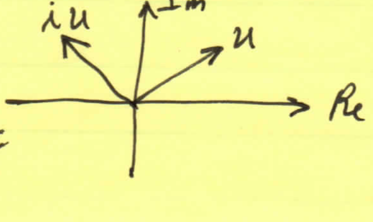
$\rho^k_\theta \cdot z = e^{ki\theta} z$, $k = 1, 2, \ldots$ is irreducible

Pf. Let $U \overset{\{0\}}{\neq} \subset \mathbb{C}$ be a S^1-invariant subspace ($\dim U = 1$)

Suppose $u \neq 0 \in U$

$\exists \theta$ st. $\rho^k_\theta u = iu$ with $\theta = \frac{\pi}{2k}$

but $iu \notin U \Rightarrow U$ is not S^1-inv. $\times\!\!\times$



e.g.) Trivial rep. on $\mathbb{R}^n$: $M_\gamma = I_n, \ \forall \gamma \in \Gamma$ is reducible for $n > 1$, since every subspace is Γ-inv.

<u>Def.</u> An action or rep. of Γ is absolutely irreducible if the only linear mappings that commute with the action of Γ on V are scalar multiples of the identity.

⊙ For rep. over $\mathbb{C}$, there is no distinction between irreducibility and absolute irreducibility. However, real rep. can be irreducible but not absolutely irreducible.

⊙ Each matrix M_{γ_i} in a reducible rep. can be decomposed as

$$M_{\gamma_i} = \begin{bmatrix} A_i & B_i \\ 0 & C_i \end{bmatrix},$$

where A_i acts on the nontrivial Γ-inv subspace. If the rep. is unitary (orthogonal) then $B_i = 0$.

Irred. rep. form the basic building blocks of the red. rep.

e.g.) $\Gamma = SO(2)$ on $\mathbb{R}^2$, standard action.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Recall: only $SO(2)$-inv. subspaces are $\{0\}$ and $\mathbb{R}^2$, so this action is irreducible. However, it is not absolutely irred. since each matrix R_θ commutes with every other rotation $R_\phi \in SO(2)$:

$$R_\theta R_\phi = R_\phi R_\theta = R_{\theta+\phi}$$

⊙ Irreducibility and absolute irreducibility are the same thing for complex representations, but not for real ones.

⊙ Absolutely irreducible Real rep. of Γ are equivalent to irred. rep. of the same dimension over $\mathbb{C}$.

(Orthogonality Thm for Matrix Rep.)

Thm Let M_γ^p and M_γ^q belong to two unitary rep. (i.e., $\underbrace{M^T = M^{-1}}_{\text{orthogonal matrices}}$ & $\det M = \{\pm 1, \pm i\}$) of Γ.

Then
$$\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\left(M_\gamma^p\right)_{jk}\left(M_\gamma^q\right)_{st} = \frac{1}{n_p}\delta_{pq}\delta_{js}\delta_{kt}$$

where $n_p \equiv \dim M^p$

e.g.) Natural rep. of D_3

$$\underbrace{\left(M_\Gamma^n\right)_{11} = \begin{array}{r} \\ \\ Me^2_{11} \\ \\ \\ \\ \end{array}\begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\begin{array}{l} Me_{11} \\ Me_{11} \\ \\ M\kappa_{11} \\ M\kappa e_{11} \\ M\kappa e^2_{11} \end{array}, \left(M_\Gamma^n\right)_{21} = \begin{bmatrix} 0 \\ \sqrt{3}/2 \\ -\sqrt{3}/2 \\ 0 \\ \sqrt{3}/2 \\ -\sqrt{3}/2 \end{bmatrix}, \left(M_\Gamma^n\right)_{12} = \begin{bmatrix} 0 \\ -\sqrt{3}/2 \\ \sqrt{3}/2 \\ 0 \\ \sqrt{3}/2 \\ -\sqrt{3}/2 \end{bmatrix}, \left(M_\Gamma^n\right)_{22} = \begin{bmatrix} 1 \\ -1/2 \\ -1/2 \\ 1 \\ -1/2 \\ -1/2 \end{bmatrix}}_{\text{orthogonal}}$$

$$\left(M_\Gamma^n\right)_{ij}\cdot\left(M_\Gamma\right)_{ij} = 3 = \frac{|\Gamma|}{n_n} = \frac{\boxed{6}\ \text{order of }\Gamma}{\boxed{2}\ \text{dim of irrep.}}$$

Identity rep. of D_3

$$\left(M_\Gamma^i\right)_{11} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

⊙ $\left(M_\Gamma^i\right)_{11} \perp \left(M_\Gamma^n\right)_{ij}$

$$\left(M_\Gamma^i\right)_{11}\cdot\left(M_\Gamma^i\right)_{11} = 6 = \frac{|\Gamma|}{n_i} = \frac{6}{1}$$