Characters

<u>Def.</u> $\chi(M) = \mathrm{trace}(M) = \sum_{i=1}^{n} M_{ii}$

◎ Let $h_1 \sim h_2$ in Γ

$$\begin{array}{ccc} h_1 & \sim & h_2 \\ | & & | \\ M_1 & & M_2 \end{array}$$

then $\chi(M_1) = \chi(M_2)$

e.g.) D_3

$M_e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $M_\rho = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$, $M_{\rho^2} = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$

$M_\kappa = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $M_{\kappa\rho} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$, $M_{\kappa\rho^2} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$

$\chi_e = 2, \chi_\rho = -1, \chi_{\rho^2} = -1$

$\chi_\kappa = 0, \chi_{\kappa\rho} = 0, \chi_{\kappa\rho^2} = 0$

◎ For finite groups, the characters of irreps specify the irreps up to equivalence

Thm The no. of inequivalent irreps of finite Γ is equal to the number of conjugacy classes of Γ

conjugacy classes

irrep	CS_1	CS_2	$\ldots$	CS_n
1	χ_{11}	χ_{12}	$\ldots$	χ_{1n}
2	$\vdots$			$\vdots$
$\vdots$				
n	χ_{n1}	χ_{n2}	$\ldots$	χ_{nn}

Character Table

Thm Let d_i = dim of the n inequivalent irreps of Γ

Then $$\sum_{i=1}^{n} d_i^2 = |\Gamma|$$

e.g. $\Gamma = D_3$

dim	irrep	$cs_1 = \{e\}$	$cs_2 = \{\rho, \rho^2\}$	$cs_3 = \{\kappa, \kappa\rho, \kappa\rho^2\}$
1	i	1	1	1
1	a	1	1	-1
2	n	2	-1	0

⑥ $1^2 + 1^2 + 2^2 = 6 = |D_3|$

trivial rep. alternating rep. natural rep.

Isotypic Decomposition

Def Let Γ be a compact Lie group acting on a vector space V. Then there exists Γ-irred subspaces U_{i}'s of V s.t.

$$V = U_1 \oplus U_2 \oplus \dots \oplus U_m$$

e.g. $\Gamma = O(2)$ acting on $V = \mathbb{R}^3$

$$\theta \in SO(2): \quad \theta \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left\{ R_\theta \begin{bmatrix} x \\ y \end{bmatrix}, z \right\}$$

$$\kappa \in O(2): \quad \kappa \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -y \\ -z \end{bmatrix}$$

$$\mathbb{R}^3 = V_1 \oplus V_2$$

$V_1 = \mathbb{R}^2 \times \{0\} = \begin{bmatrix} x \\ y \end{bmatrix}$ } standard irrep of $O(2)$

$V_2 = \{0\} \times \{\mathbb{R}\} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$ } nontrivial 1D irrep of $O(2)$

$\Theta\ U_1 \oplus \dots \oplus U_m$ is not unique because some U_i's can be Γ-isomorphic to each other.

Sol Let $W_1 = \underbrace{U^{(1)} \oplus U^{(2)} \oplus \dots \oplus U^{(m_1)}}_{\text{isomorphic}}$

$W_2 = U^{(1)} \oplus U^{(2)} \oplus \dots \oplus U^{(n_2)}$

$\vdots$

W_k

Isotypic Decomp. of V

$$\Rightarrow \boxed{V = W_1 \oplus W_2 \oplus \dots \oplus W_k, \quad k \leq m}$$

isotypic components

e.g.) $\Gamma = SO(2)$, $V = \mathbb{R}^{2\times 2}$

$$\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Any 2x2 matrix can be written as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}}_{V_1} \oplus \underbrace{\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}}_{V_2}$$

$$\left.\begin{aligned} \theta_1 \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} &= \begin{pmatrix} a \\ c \end{pmatrix} \in \mathbb{R}^2. \\ \theta_2 \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} &= \begin{pmatrix} b \\ d \end{pmatrix} \in \mathbb{R}^2 \end{aligned}\right\} \quad \theta_{1,2} : V_{1,2} \longrightarrow \mathbb{R}^2$$

Remarks

1. Action of $SO(2)$ on V_1 & V_2 is isomorphic to its standard irred. matrix-multiplication action on $\mathbb{R}^2$ via

$$\Theta_{1,2} : V_{1,2} \longrightarrow \mathbb{R}^2 \Rightarrow V_1 \overset{SO(2)\text{-iso}}{\sim} V_2$$

2. Isotypic decomposition $W = V = \mathbb{R}^2$ is unique as it consists of one component, the whole $\mathbb{R}^2$ space.

3. But the decomposition of V into irred. subspaces is not unique since $so(2)$ also acts irred. on

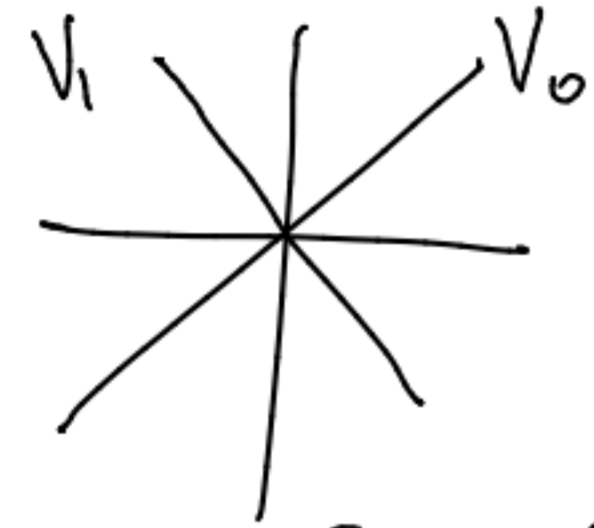
$$V_3 = \begin{pmatrix} b & b \\ d & d \end{pmatrix}$$

4. Summary:

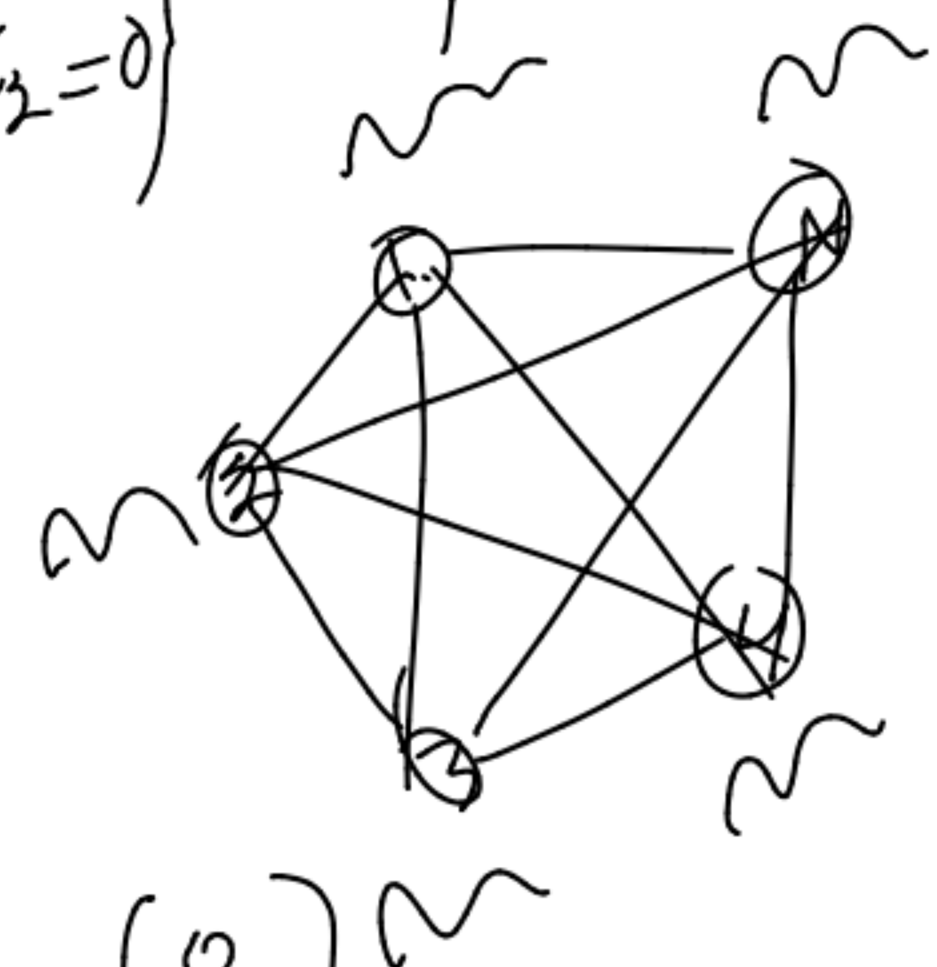
$$V = V_1 \oplus V_3 \quad \text{or} \quad V = V_1 \oplus V_2$$

e.g.) $\Gamma - S_n$ acting on $\mathbb{R}^n$

$n=2$ $\quad V_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$V_1 = \{x_1 + x_2 = 0\} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



Let

$$V_0 = \mathbb{R}\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{n \text{ times}} \right\}$$

$$V_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_1 + x_2 + \dots + x_n = 0 \right\}$$

$x_1 + x_2 + x_3 = 0$

$x_3 = -x_1 - x_2$

$x_n = -x_1 - x_2 - \dots - x_{n-1}$: $\begin{bmatrix} x_1 \\ x_2 \\ -x_1 - x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$W_0 \oplus W_1 \oplus \dots \oplus W_k$