Ch 4 ODEs with Symmetry

(ODE) $\dfrac{dx}{dt} = f(x, \mu)$, $x \in \mathbb{R}^n$, $\underbrace{\mu \in \mathbb{R}}_{\text{bif. par.}}$

Let $\gamma \in \Gamma \equiv$ symmetry group of bif. problem

$x(t) \equiv$ sol. of (ODE)

Then $\gamma x(t)$ is also a sol. for all $\gamma \in \Gamma$

Let $y(t) = \gamma x(t)$, then

$$\frac{dy}{dt} = f(y(t), \mu) = f(\gamma x, \mu)$$

$$\overset{\text{want}}{=}$$

$$\cancel{\gamma}\frac{dx}{dt} = \cancel{\gamma} f(x, \mu)$$

$\Rightarrow f(x, \mu)$ must satisfy

$$\boxed{\gamma f(x, \mu) = f(\gamma x, \mu)}$$

$\forall \gamma \in \Gamma$ Equivariant Condition

<u>Thm</u> Γ is a symmetry group of (ODE) iff

$$f(\gamma x, \mu) = \gamma f(x, \mu)$$

$\Rightarrow f$ commutes with Γ

$\Rightarrow M_\gamma f(x, \mu) = f(M_\gamma x, \mu)$

Recall, bif. cond.

$$\frac{dx}{dt} = f(x, \mu) \quad (ODE)$$

$$Df\Big|_{(x_0,\mu_0)} = (df)_{(x_0,\mu)} \equiv \text{Jacobian Matrix}$$

$$\text{e-val}\{(df)_{(x_0,\mu_0)}\} = \{0, \ldots\}$$

$$\overset{??}{(0,0)}$$

Assume f to be Γ-equiv.

$$f(\gamma x) = \gamma f(x), \quad \forall \gamma \in \Gamma$$

$\Downarrow$ diff. & apply chain role

$$(df)_{(\gamma x)}\gamma = \gamma (df)_{(x)}$$

$$\Rightarrow \boxed{(df)_{(0)}\gamma = \gamma (df)_{(0)}}$$

Jacobian (df) is Γ-equiv. at equilibrium $(0,0)$.

Group Orbits

Let x = equil. sol.

Def $\Gamma x = \{\gamma x : \gamma \in \Gamma\}$

orbit of Γ on $x \in V$

⓪ x need not be a sol.

Remarks

1. If v is an e-vec with 0 e-val so is any point γv in the orbit of v
2. Symmetric ODEs usually have multiple zero e-vals.

Claim

If v is an e-vec of $(df)_x$ with e-val λ then

γv is an e-vec of $(df)_{(\gamma x)}$ with e-val λ.

Pf. $(df)_x v = \lambda v$

$$\Rightarrow (df)_{(\gamma x)} \gamma v = \gamma (df)_{(x)} v = \gamma(\lambda v)$$
$$= \lambda(\gamma v)$$

⓪ If $(df)_{(0,\mu)} v = 0$ then $(df)_{(0,\mu)} \gamma v = 0$

$\square$

Isotropy Subgroups

Def The symmetry of a stationary sol. x is given by

$$\Sigma_x = \left\{ \sigma \in \Gamma : \sigma x = x \right\}$$

Lemma: Points on the same orbit of Γ have conjugate isotropy subgroups:

$$\Sigma_{\gamma x} = \gamma \Sigma_x \gamma^{-1}$$

Fact: $(df)_{(x,\mu)}$ commutes with all elements of Σ_x

Pf Let $\gamma = \sigma \in \Sigma_x$

Recall: $\gamma (df)_x = (df)_{(\gamma x)} \gamma$

$\Rightarrow \sigma (df)_{(x,\mu)} = (df)_{(\sigma x),\mu)} \sigma = (df)_{(x),\mu)} \sigma$ $\quad \sigma x \in \Sigma_x$.

$$\boxed{\Rightarrow (df)\sigma y = \sigma (df) y = 0, \quad \forall y \in \ker(df)}$$

$\Rightarrow$ If $y \in \ker(df)$ so is σy

hm Let $V = W_1 \oplus \cdots \oplus W_s$
e a rep. of Γ and
W_k's is a isotypic decomp.
et $A : V \longrightarrow V$ be linear and commuting

hen $\underbrace{A(W_k) \subset W_k}_{\text{inv. of } W_k\text{'s under } A}$

$$(df)V = \begin{bmatrix} (df)W_1 & & & \\ & (df)W_2 & & \\ & & \ddots & \\ & & & (df)W_s \end{bmatrix}$$

$$W_1 = \begin{pmatrix} \vdots \end{pmatrix}$$

Def The fixed-point subspace of a subgroup Σ is:

$$\mathrm{Fix}(\Sigma) = \{ x \in V : \sigma x = x, \ \forall \sigma \in \Sigma \}$$

Since σ is linear then $\sigma \cdot 0 = 0 \Rightarrow \mathrm{Fix}(\Sigma)$ always contains $\{0\}$.

$$\frac{dx}{dt} = f(x, \mu)$$

SS $f(x, \mu) = 0$

Idea: $f(x, \mu)\Big|_{\mathrm{Fix}(\Sigma)} = 0$

Thm Fixed-point subspaces are flow inv.

Pf Let $x \in \mathrm{Fix}(\Sigma)$

$$\Rightarrow f(x) = f(\sigma x) = \sigma \cdot f(x)$$

$$\Rightarrow \sigma f = f$$

$$\Rightarrow f(\mathrm{Fix}(\Sigma)) \subset \mathrm{Fix}(\Sigma)$$

Thm **Equivariant Branching Lemma**

Let (i) Γ = finite or compact Lie group acting abs. irreducible on a real vector space V

(ii) $\frac{dx}{dt} = f(x, \mu)$ be Γ-equiv.

Bif. Cond.

(iii) $\boxed{\begin{aligned} &f(0,0) = 0 \\ &(df)_{(0,0)} = 0 \end{aligned}}$ — equil. cond / — zero e-val cond

(iv) $\dim \mathrm{Fix}(\Sigma) = 1$, for some isotropy subgroup $\Sigma \subset \Gamma$

Then, $\exists$ a unique branch of sol. to $f(x, \mu) = 0$ st. each sol. on this branch has isotropy subgroup Σ