Summary

$$\frac{dx}{dt} = f(x,\mu), \quad x \in \mathbb{R}^n, \mu \in \mathbb{R}$$

Γ-symmetry: $\underbrace{f(\gamma x) = \gamma f(x)}_{\Gamma\text{-equiv}}, \ \forall \gamma \in \Gamma$

$$Df\big|_{(x_0,\mu_0)} = (df)_{(x_0,\mu_0)}$$

$$(df)_{\gamma x}\gamma = \gamma (df)_x$$

- Isotropy Subgroups: $\Sigma_x = \{\sigma \in \Gamma : \sigma x = x\}$
- If $y \in \ker(df)$ so is $\sigma y \Rightarrow \ker(df)$ is Σ_x-inv.
- $V = W_1 \oplus \cdots \oplus W_s$, $A(W_k) \subset W_k$

$$(df) = \begin{bmatrix} (df)W_1 & & \\ & \ddots & \\ & & (df)W_s \end{bmatrix}$$

- $\mathrm{Fix}\,\Sigma = \{x \in V : \sigma x = x\}$
- $f : \mathrm{fix}(\Sigma) \to \mathrm{fix}(\Sigma)$
- EBL

$f(0,0) = 0$, $(df)_{(0,0)} = 0$

$\dim \mathrm{Fix}(\Sigma) = 1$

$\Rightarrow \exists$ unique branch of sol. to $f(x,\mu) = 0$ with Σ_x symmetry.

- The following items are equiv.

(i) $\mathrm{Fix}(\Gamma) = \{0\}$

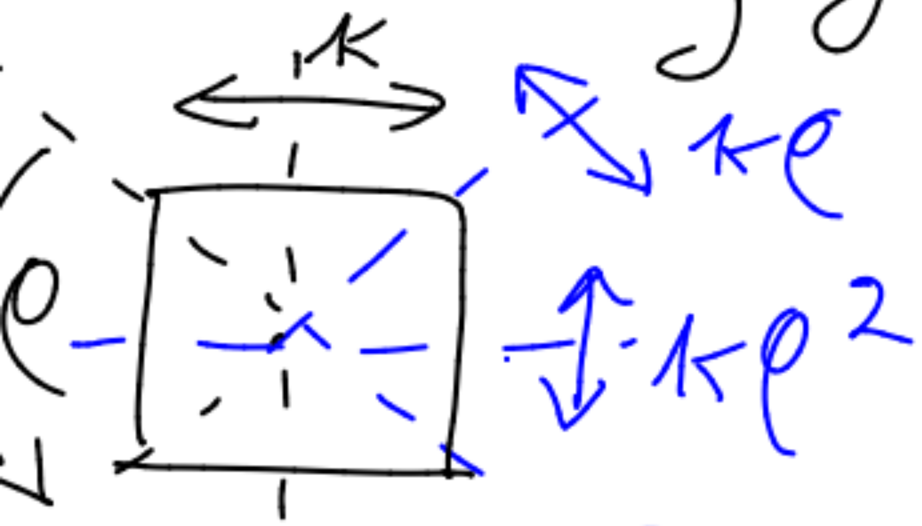
(ii) $f(0,0) = 0$

(iii) The only Γ-inv linear function is zero.

Example: Steady-State Bif.

in a square box: $\Gamma = D_4$

Pattern-Forming System with D_4-sym



$\rho = e^{\frac{2\pi}{4} i}$

Irrep

		e	ρ	ρ^2	ρ^3	κ	$\kappa\rho$	$\kappa\rho^2$	$\kappa\rho^3$
abs irrep (1D)	R_1	1	1	1	1	1	1	1	1
	R_2	1	1	1	1	-1	-1	-1	-1
	R_3	1	-1	1	-1	1	-1	1	-1
	R_4	1	-1	1	-1	-1	1	-1	1

R_5 (2D)

$M_e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M_\rho = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, M_{\rho^2} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, M_{\rho^3} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$M_\kappa = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, M_{\kappa\rho} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, M_{\kappa\rho^2} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, M_{\kappa\rho^3} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

S2 Each irrep leads to a bif. problem

$$\dot{x} = f(x,\mu), \quad x \in \mathbb{R}^n$$

where $f(\gamma x, \mu) = \gamma f(x,\mu), \ \forall \gamma \in D_4$

⊙ x is the amplitude of an e-mode represented by the corresponding basis vector

$$u(\cancel{y}, t) = \sum_{k=1}^{M} x(t)\,\Phi(\cancel{y})$$

S3 Find $f(x,\mu)$

Assume: $f(0,0) = 0$, $(df)_{(0,0)} = 0$

ct $K = \ker(df)_{(0,0)} \neq 0$

Ve know: $(df)_{(0,0)}\gamma = \gamma(df)_{(0,0)}$

$\forall \gamma \in D_4$

$x \in \mathbb{R}^1$, $\mathrm{Fix}(D_4) = \mathbb{R}$

R_1 acts abs. irred. but

$\mathrm{Fix}(D_4) = \mathbb{R}$ rather than $\{0\}$

$\Rightarrow$ Can't apply EBL

leed: $M f(x,\mu) = f(Mx,\mu)$

$M_s = 1$, $1 \cdot f(x,\mu) = f(1\cdot x, \mu)$

$\Rightarrow$ NO RESTRICTION ON f

⊙ Still need: $f(0,0)=0$, $(df)_{(0,0)} = 0$

$$\frac{dx}{dt} = \mu + \underbrace{a}_{}x^2 + \underbrace{b}_{}x\mu + \underbrace{c}_{}\mu^2 + \dots \qquad (a \neq 0)$$

constants

Near $\mu \sim 0$, $x \sim \mu^{1/2}$, $x\mu, \mu^2$ are h.o.t.

$$\Rightarrow \boxed{\frac{dx}{dt} = \mu + ax^2 + \text{h.o.t.}}$$

Saddle-Node Bif.

⊙ Pattern has D_4-Symmetry

S5

R_2, B_3, R_4

$M_S = +1: \quad 1\cdot f(x,\mu) = f(1\cdot x,\mu) \Rightarrow$ no restriction

$M_S = -1: \quad -1\cdot f(x,\mu) = f(-1\cdot x,\mu) \Rightarrow$ f must be odd in x

$$\Rightarrow \boxed{\frac{dx}{dt} = \mu x + a x^3 + \text{h.o.t.}}$$

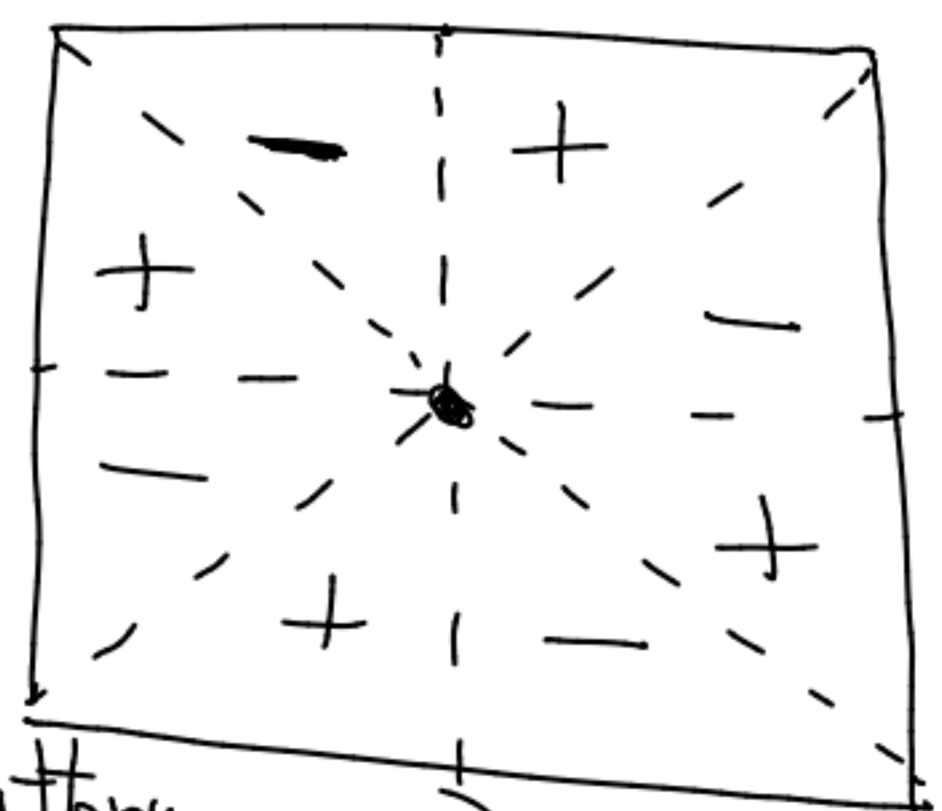
Pitchfork Bif.

Subcritical if $a>0$, Supercritical if $a<0$

Eigenmodes

$A_1 = \{e, \rho, \rho^2, \rho^3\}$

$A_1 = \{\kappa, \kappa\rho, \kappa\rho^2, \kappa\rho^3\}$



Pattern_{R_2} $\quad R_2$

$$\Sigma_{\text{Pattern}_2} = \{e, \rho, \rho^2, \rho^3\} = \mathbb{Z}_4$$

$$\text{Fix}(D_4) = \{0\}$$

$$\underbrace{\dim(\text{Fix}(\mathbb{Z}_4)) = 1 \text{ (since there is only one deg. of freedom, } x)}$$

$\Downarrow$ Can apply EBL

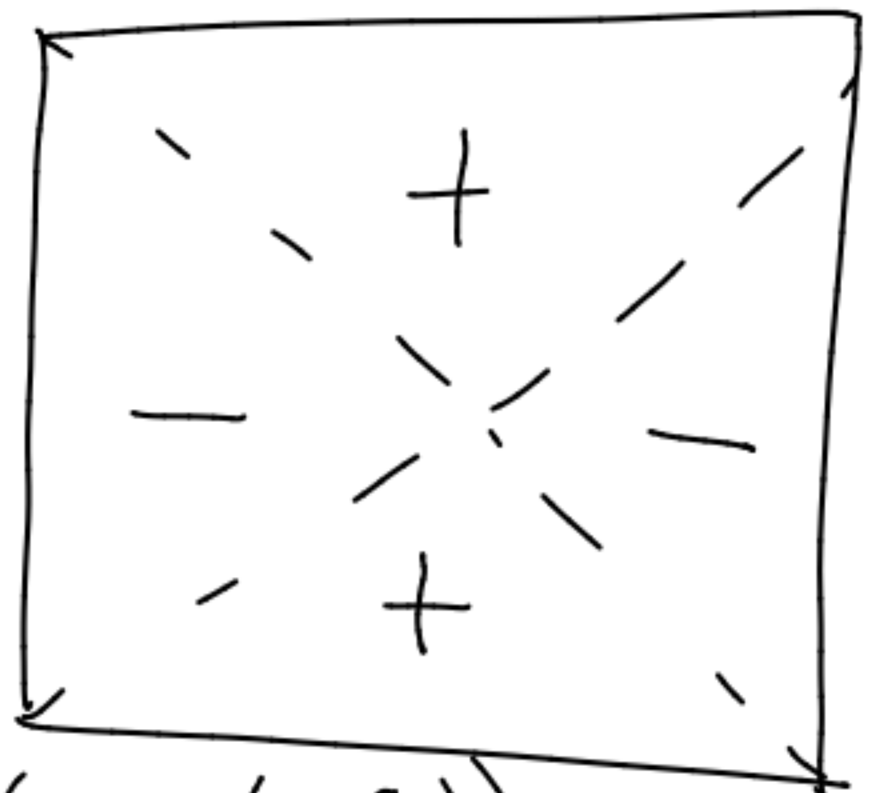
$\exists$ a branch of patterns Pattern_{R_2} with isotropy subgroup $\mathbb{Z}_4$, which bif. from $x=0$ at $\mu=0$.

$\underline{\ 3\ }$

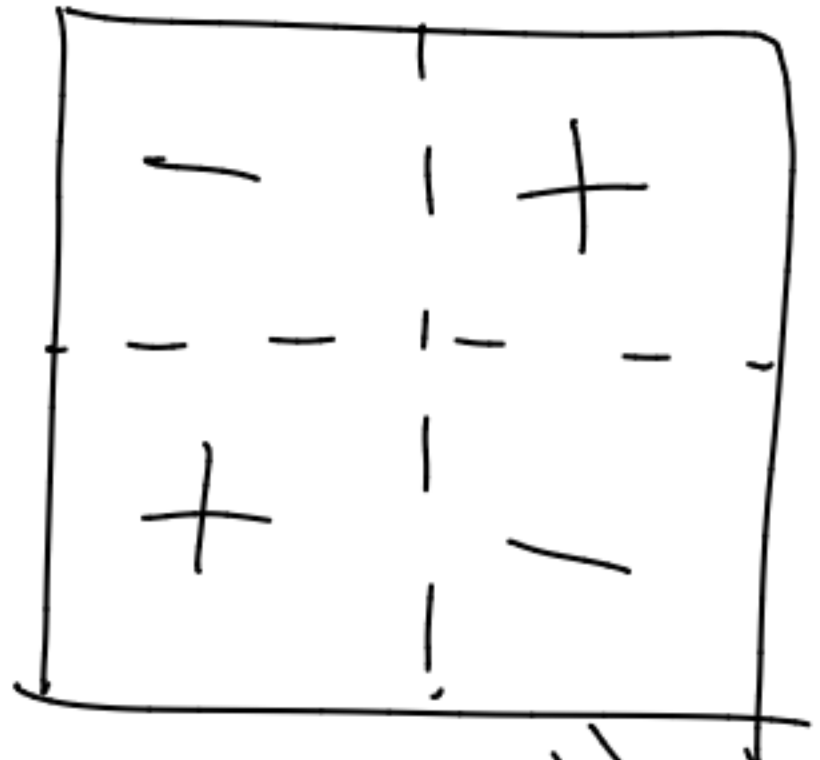
$$\Sigma_{P_{R_3}} = \{e, \rho^2, \kappa, \kappa\rho^2\} = \mathbb{Z}_2^2$$

$\underline{R_4}$

$$\Sigma_{P_{R_4}} = \{e, \rho^2, \kappa\rho, \kappa\rho^3\} = \mathbb{Z}_2^2$$



$$\dim(\mathrm{Fix}(\mathbb{Z}_2^2)) = 1$$



$$\dim(\mathrm{Fix}(\mathbb{Z}_2^2)) = 1$$

$$\begin{array}{c} D_4 \\ \uparrow \\ \mathbb{Z}_2^2 \end{array}$$