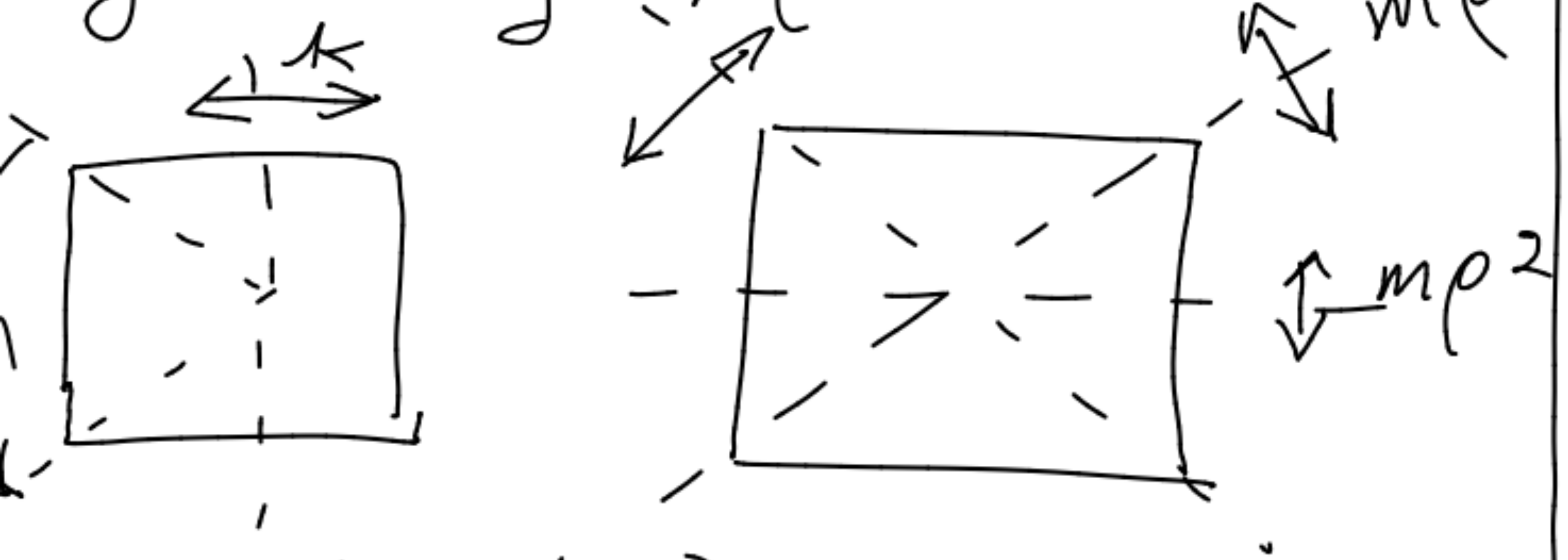
Pattern Formation with $\Gamma = D_4$ symmetry



Natural Rep.

$$M_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_\rho = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, M_{\rho^2} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, M_{\rho^3} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$M_\kappa = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, M_{\kappa\rho} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, M_{\kappa\rho^2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, M_{\kappa\rho^3} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Let $x \in \mathbb{R}^2$

$$\frac{dx}{dt} = f(x, \mu) \iff \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, \mu) \\ f_2(x_1, x_2, \mu) \end{bmatrix}$$

$$M_\kappa = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f(M_\gamma x) = M_\gamma f(x)$$

$$M_\kappa \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} \Rightarrow f(M_\kappa x) = \begin{pmatrix} f_1(-x_1, x_2) \\ f_2(-x_1, x_2) \end{pmatrix}$$

$$M_\kappa \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -f_1 \\ f_2 \end{pmatrix}$$

want

$\Rightarrow$ Equivariant Condition:

$$\begin{bmatrix} -f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} f_1(-x_1, x_2) \\ f_2(-x_1, x_2) \end{bmatrix}$$

$\Rightarrow$ f_1 must be odd in x_1
f_2 " even in x_1

$$M_e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \underbrace{\begin{bmatrix} -f_2(x_1, x_2) \\ \boxed{f_1(x_1, x_2)} \end{bmatrix}}_{M_e f} = \underbrace{\begin{bmatrix} f_1(-x_2, x_1) \\ \boxed{f_2(-x_2, x_1)} \end{bmatrix}}_{f(M_e x)}$$

$$f_1(x_1, -x_2) = f(x_2, x_1)$$

$$f_1(x_1, x_2) = \underbrace{f_2(-x_2, x_1) \underset{M_K}{\overset{\text{from}}{=}} f_2(x_2, x_1)}_{f_2 \text{ is even in } 1^{st} \text{ component}} = f_1(x_1, -x_2)$$

$$\Downarrow$$

f_1 is even in 2^{nd} component

Then,

$$f_1(x_1, x_2, \mu) = \mu x_1 - a_1 x_1^3 - a_2 x_2^2 x_1 + h.o.t.$$

$$\Rightarrow f_1(-x_2, x_1, \mu) = \mu(-x_2) - a_1(-x_2)^3 - a_2(x_1)^2(-x_2) + \ldots,$$

$$-f_1(-x_2, x_1, \mu) = f_2(x_1, x_2)$$

$$\Rightarrow f_2(x_1, x_2) = \mu x_2 - a_1 x_2^3 - a_2 x_1^2 x_2 + \dots$$

Then,

$$f_1(x_1, x_2) = \mu x_1 - a_1 x_1^3 - a_2 x_2^2 x_1 + \dots$$

$$f_2(x_1, x_2) = \mu x_2 - a_1 x_2^3 - a_2 x_1^2 x_2 + \dots$$

$$\Rightarrow \boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \mu \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - a_1 \begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix} - a_2 \begin{bmatrix} x_2^2 x_1 \\ x_1^2 x_2 \end{bmatrix}}$$

$$\Rightarrow \boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \mu \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \substack{a_1 > 0 \\ - \\ + \\ a_1 < 0} \begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix} - a_2 \begin{bmatrix} x_2^2 x_1 \\ x_1^2 x_2 \end{bmatrix}}$$

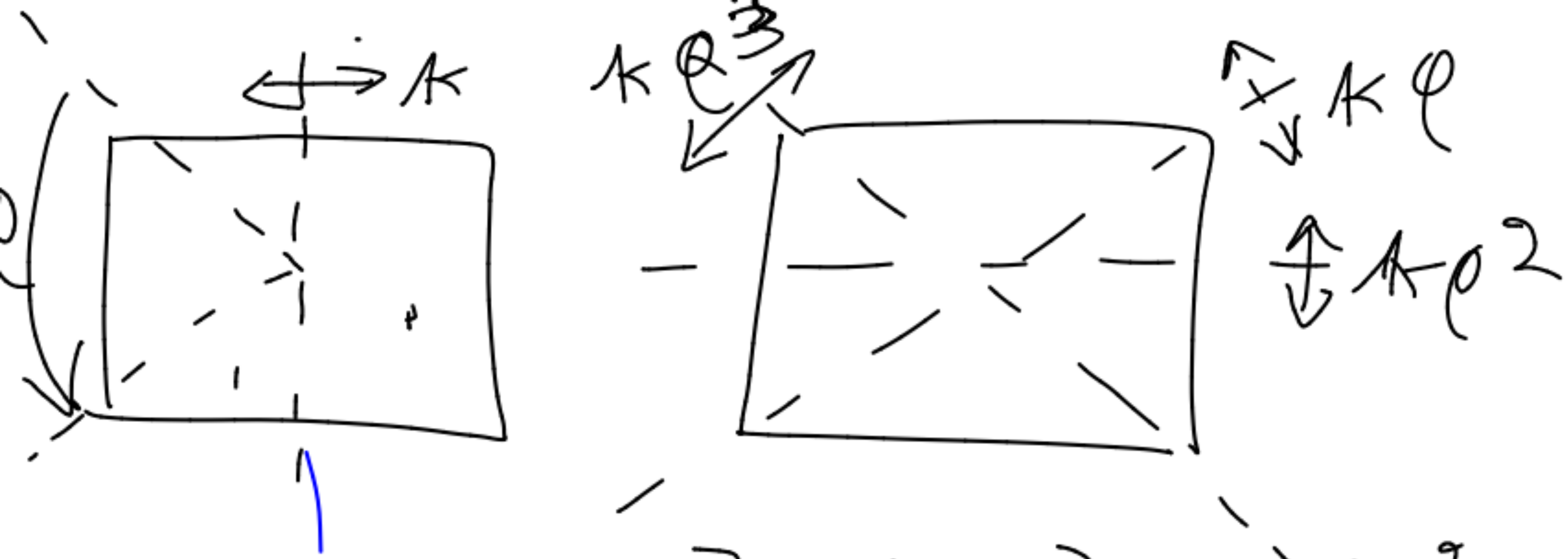
$$\hat{x}_i = \sqrt{a_1}\, x_i, \quad i = 1, 2$$

$$\hat{a}_2 = \frac{a_2}{|a_1|}$$

Rescaling with $a_1 = \pm 1$

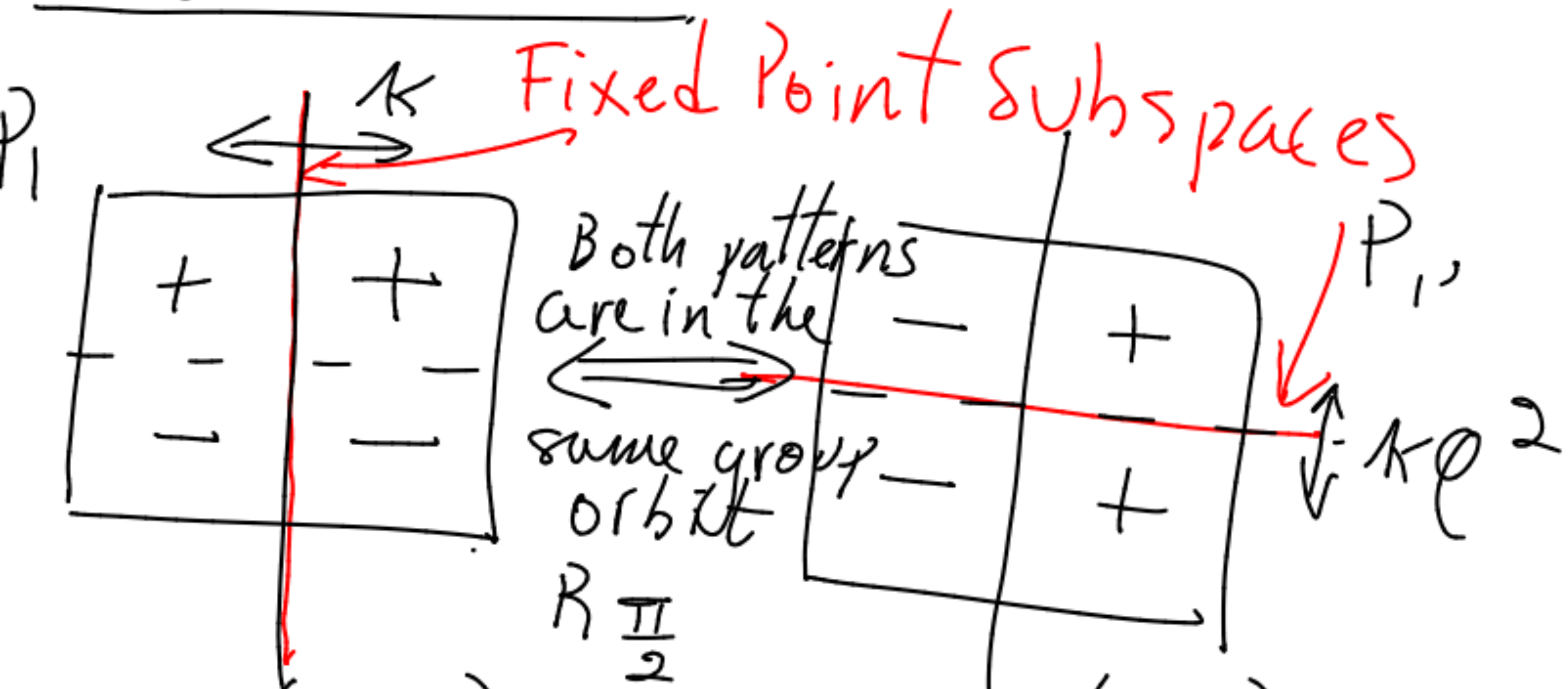
Amplitude Equations

(Exercise: Solve in MATLAB)



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \mu \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mp \begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix} - a_2 \begin{bmatrix} x_2^2 x_1 \\ x_1^2 x_2 \end{bmatrix}$$

Eigenmodes



F.P. Subspace

$P_1 : \{(0, x_2)\}$

F.P. Subspace

$P_{1'} : \{(x_1, 0)\}$

$$M_\kappa = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad M_{\kappa\rho^2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Downarrow$$

$$\Sigma = \mathbb{Z}_2(\kappa) = \{e, \kappa\}, \quad \Sigma = \mathbb{Z}_2(\kappa) = \{e, \kappa\rho^2\} \quad \dim(F.P.(P_1, P_{1'})) = 1$$

Same stability properties

$$\Sigma_{P_1} = \mathbb{Z}_2(k)$$

$$F.P.(P_1) = \{(x_1, 0)\} = \mathbb{R}^1$$

$$\dim(F.P.(\mathbb{Z}_2(k))) = 1$$

Set $x_2 = 0$

$$\frac{dx_1}{dt} = \mu x_1 \mp x_1^3$$

$$x_1 = \pm\sqrt{\frac{\mu}{a_1}}$$

$$\Sigma_{P_{1'}} = \mathbb{Z}_2(k)$$

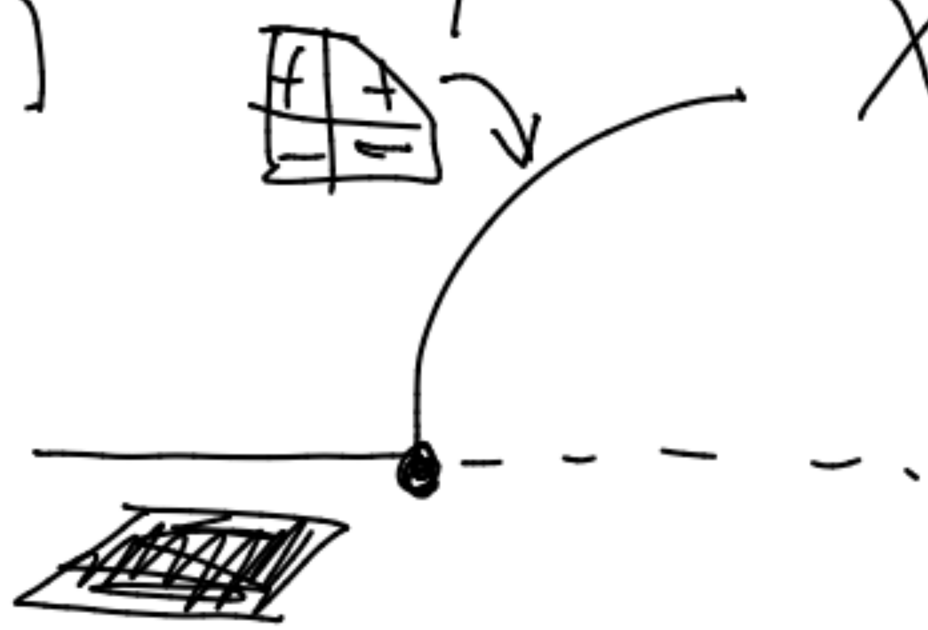
$$F.P.(P_{1'}) = \{(0, x_2)\} = \mathbb{R}^1$$

$$\dim(F.P.(\mathbb{Z}_2(k))) = 1$$

Set $x_1 = 0$

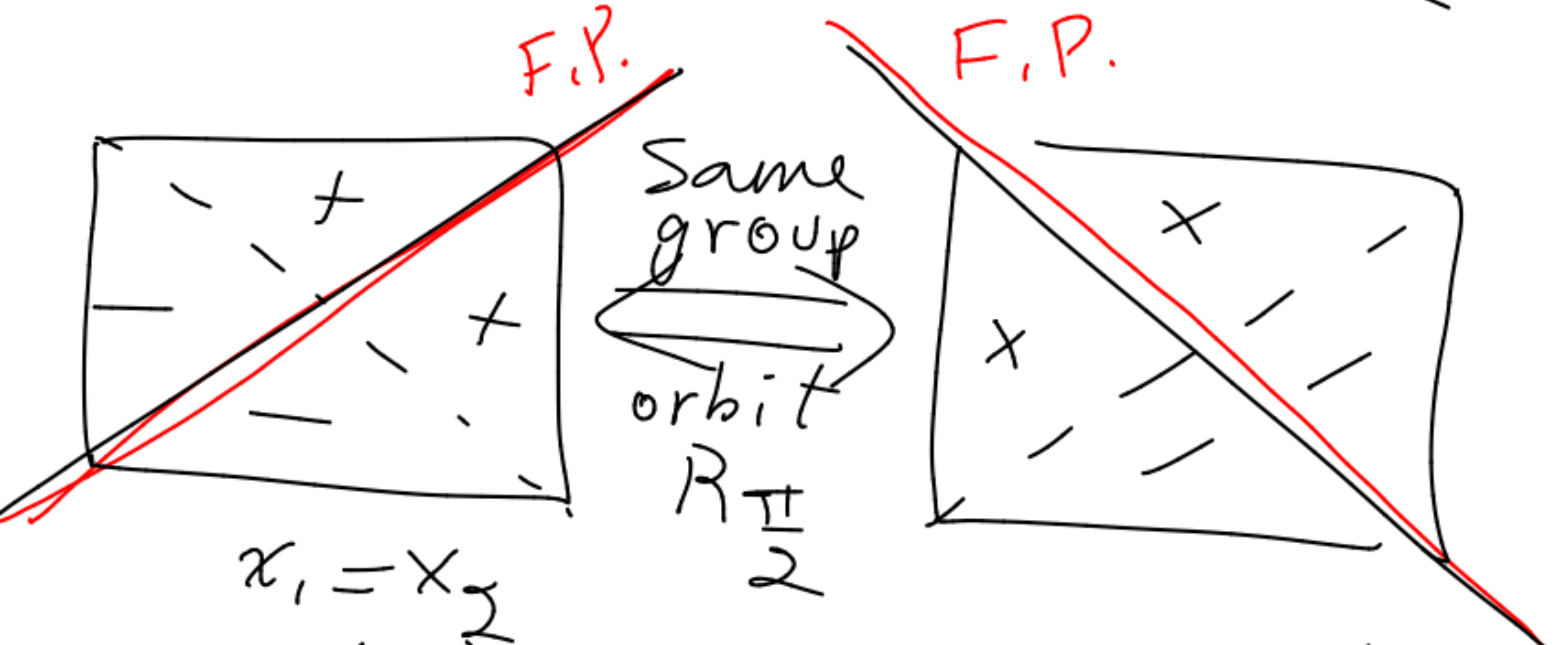
$$\frac{dx_2}{dt} = \mu x_2 \mp x_2^3$$

$$x_2 = \pm\sqrt{\frac{\mu}{a_1}}$$



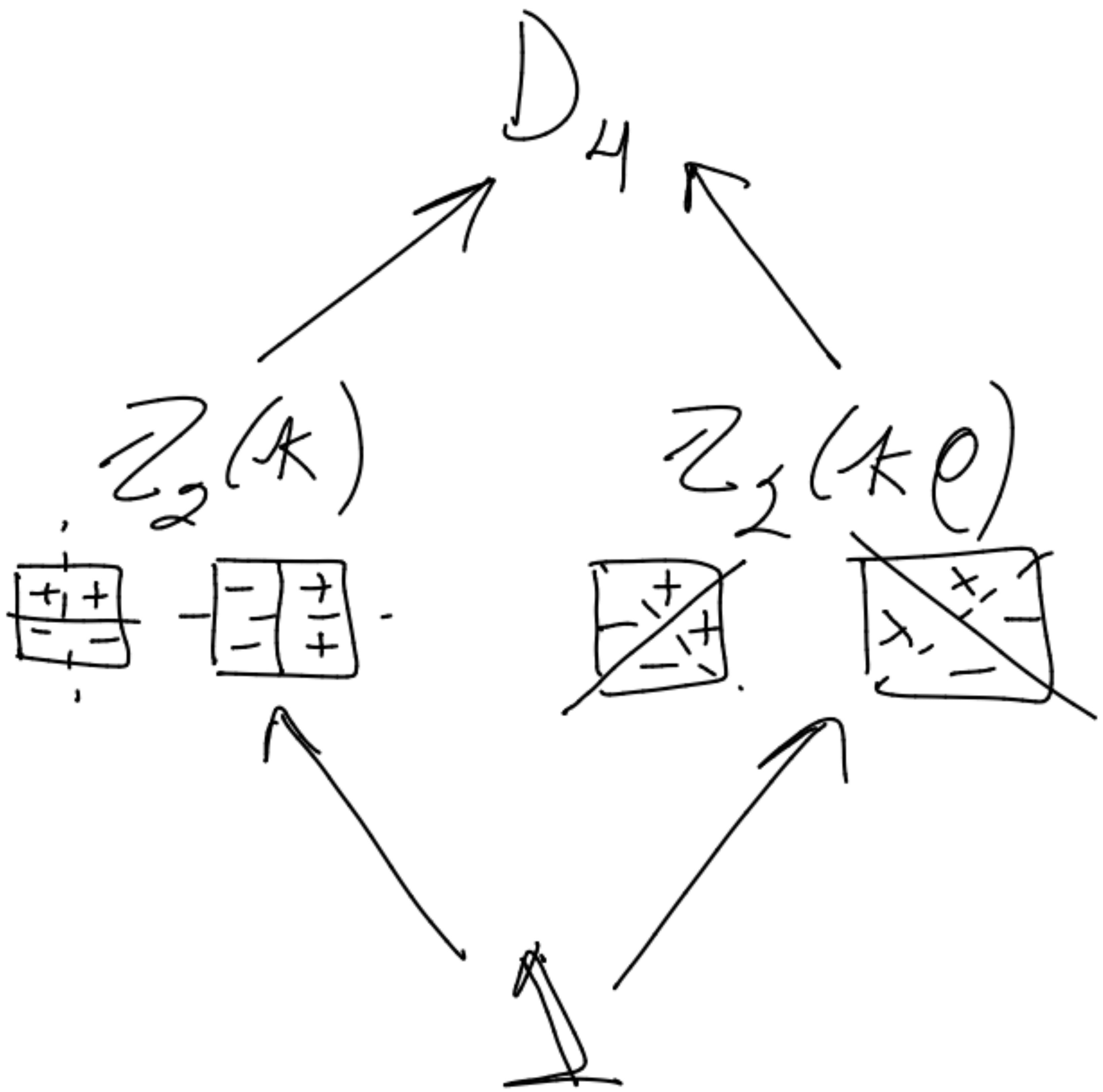
Mixed-Mode Sol

Set $x_1 = x_2$ and $x_1 = -x_2$



$$\Sigma = \mathbb{Z}_2(\kappa\rho) = \{e, \kappa\rho\} \qquad \Sigma = \mathbb{Z}_2(\kappa\rho^3) = \{e, \kappa\rho^3\}$$

Lattice of Isot—y Subgroups



Stability

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \mu \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mp \begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix} - a_2 \begin{bmatrix} x_2^2 x_1 \\ x_1^2 x_2 \end{bmatrix}$$

Jacobian :

$$Df = \begin{bmatrix} \mu - 3a_1 x_1^2 - a_2 x_2^2 & -2a_2 x_1 x_2 \\ -2a_2 x_1 x_2 & \mu - 3a_1 x_2^2 - a_2 x_1^2 \end{bmatrix}$$

$P_1\ (x_1, 0)$, $x_1^2 = \mu/a_1$

$$Df\Big|_{P_1} = \begin{bmatrix} -2\mu & 0 \\ 0 & \mu\left(1-\frac{a_2}{a_1}\right) \end{bmatrix}$$

e-vals: $\sigma_1 = -2\mu$

$\sigma_2 = \mu\left(1 - \frac{a_2}{a_1}\right)$

$\Rightarrow$ P_1 is stable (in x_1-dir) if $\mu > 0$ and in x_2-dir if $a_1 < a_2$

$\Sigma_{P_1} = \{e, \kappa\rho^2\}$

Isotypic Components:

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ b \end{bmatrix} \right\}$$

$\Downarrow$

Df is diagonal

Pattern $P_3(x_1, x_1)$, $x_1^2 = \frac{\mu}{a_1+a_2}$.

$$Df\Big|_{P_3} = \begin{bmatrix} -\frac{2a_1\mu}{a_1+a_2} & -\frac{2a_2\mu}{a_1+a_2} \\ -\frac{2a_2\mu}{a_1+a_2} & -\frac{2a_1\mu}{a_1+a_2} \end{bmatrix}$$

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} a \\ a \end{bmatrix}, \begin{bmatrix} b \\ -b \end{bmatrix} \right\} \Longrightarrow (Df)\Big|_{P_3} = \begin{bmatrix} \alpha & 0 \\ 0 & * \end{bmatrix}$$

$$\sigma_1 = -2\mu, \quad \sigma_2 = -\frac{2\mu(a_1-a_2)}{a_1+a_2}$$

Pattern $= P_0$

Trivial Sol. $x_1 = x_2 = 0$

$$Df\Big|_{P_0} = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}$$

P_0 is stable if $\mu < 0$

P_0 is unstable if $\mu > 0$