4.1

$\dot{x} = f(x, y)$
$\dot{y} = g(x, y)$

$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -y \\ x \end{bmatrix}$

Show f is odd in x, even in y
g is odd in y, even in x

$M_{k_1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \Rightarrow M_{k_1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$M_{k_2} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} \Rightarrow M_{k_2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Let $F(x,y) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$

Need to find $F(x,y)$ st.

$M_{k_1} F(x,y) = F(M_{k_1}(x,y))$
$M_{k_2} F(x,y) = F(M_{k_2}(x,y))$

$M_{k_1} F = \begin{bmatrix} -f(x,y) \\ g(x,y) \end{bmatrix}$

$F(M_{k_1}(x,y)) = \begin{bmatrix} f(-x, y) \\ g(-x, y) \end{bmatrix}$

want

$f(-x, y) = -f(x, y)$
$g(x,y) = g(-x, y)$

Finding irreps

Let $\Gamma \equiv$ group of symmetries

S1 Identify generators γ_i of Γ

S2 Assuming $\underbrace{|\gamma_i|}_{\text{order of } \gamma_i} = p \Rightarrow \gamma_i^p = e$

(*)

$$\Rightarrow \boxed{M_{\gamma_i^p} = I} \Rightarrow M_{\gamma_i} \equiv (M_{\gamma_i^p})^{1/p}$$

- Not all roots of (*) lead to irreps.

e.g.) $\Gamma = D_4$, $|D_4| = 8$

Idea: Find conjugacy classes within D_4

Recall: Let $h_1, h_2 \in D_4$

$h_1 \overset{\text{conjugate}}{\sim} h_2$ iff $\exists \gamma \in D_4$ st

$$h_1 = \gamma h_2 \gamma^{-1}$$

$$|h_1| = |h_2| \quad \text{(same order)}$$

$$\gamma e \gamma^{-1} = e, \ \forall \gamma \in D_4$$

$\Rightarrow \{e\}$ is in its own class of order 1

$$|\rho| = |\rho^3| = 4$$

$$|\rho^2| = |k| = |k\rho| = |k\rho^2| = |k\rho^3| = 2$$

Is $\rho \sim \rho^3$? Are they in the same class?

$$(k\rho)\rho^3(k\rho)^{-1} = k\underbrace{\rho^4}_{e}\rho^{-1}k^{-1} = k\rho^{-1}k$$

$$k\rho^{-n} = \rho^n k \qquad = \rho k k = \rho \ \checkmark$$

$n = 1, 2, 3$

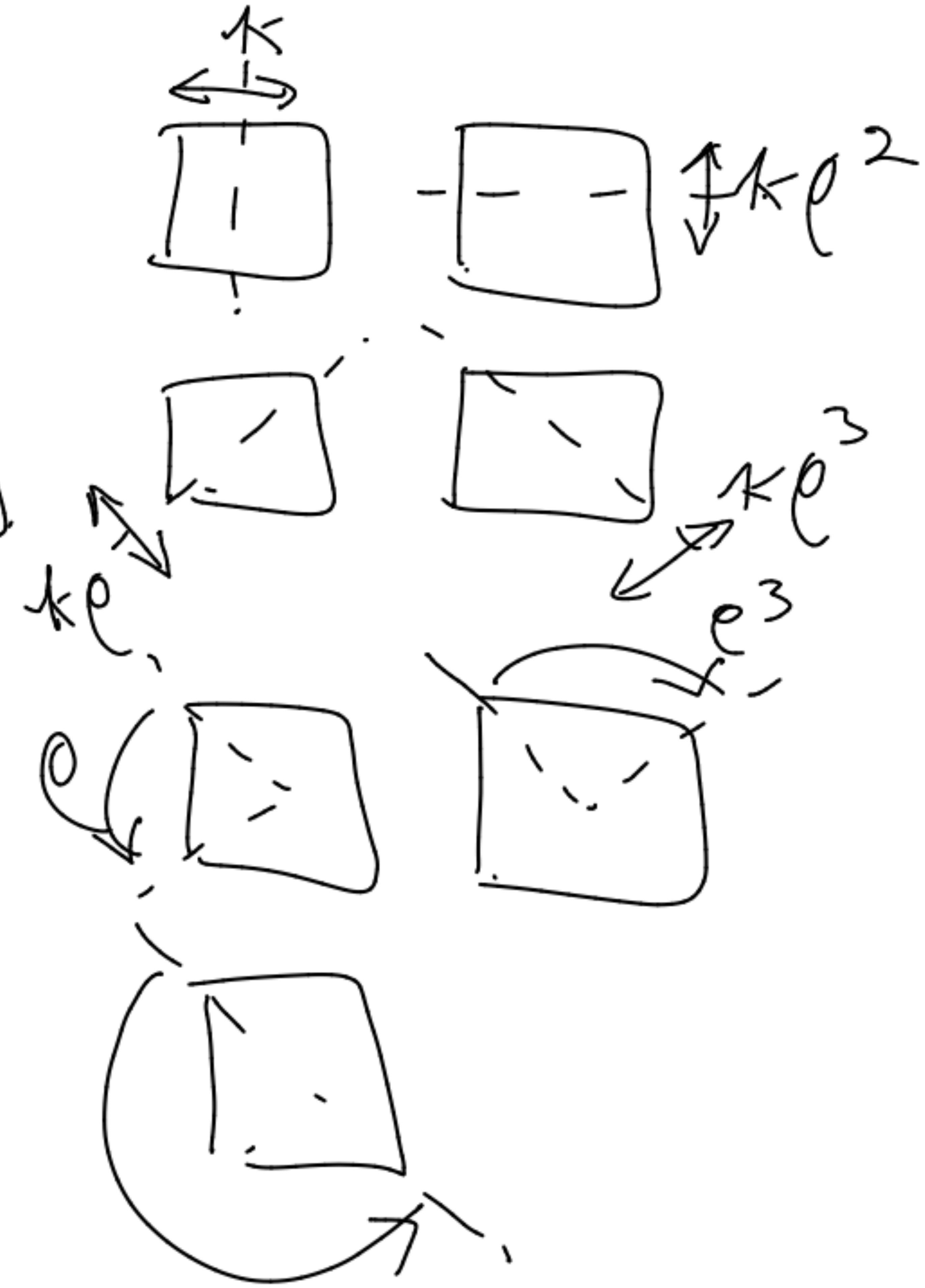
$\{e\}$

$\{k, k\rho^2\}$

$\{k\rho, k\rho^3\}$

$\{\rho, \rho^3\}$

$\{\rho^2\}$



Thm The number of inequivalent irreps of a finite group Γ is equal to the number of conjugacy classes

g.) $\Gamma = D_4 \Longrightarrow 5$ irreps

call:

Thm $\displaystyle\sum_{i=1}^{n} d_i^2 = |\Gamma|$

$|\Gamma = D_4| = 8 \Longrightarrow \displaystyle\sum_{i=1}^{n} d_i^2 = 8$

cl.

$\Longrightarrow d_1 = \dots = d_4 = 1,\ d_5 = 2$

$M_{\kappa^2} = 1,\quad M_{\rho^4} = 1$

$\Downarrow$

$M_\kappa = \pm 1,\quad M_\rho = \pm 1, \pm i$

Choose

$$\boxed{M_\kappa = -1,\quad M_\rho = 1}$$

$$M_{\kappa\rho} = M_\kappa M_\rho = -1 \times 1 = -1$$

$$M_{\rho^2} = M_\rho M_\rho = 1 \times 1 = 1$$

irrep	e	ρ	ρ^2	ρ^3	κ	$\kappa\rho$	$\kappa\rho^2$	$\kappa\rho^3$
R_1	1	1	1	1	1	1	1	1
R_2	1	1	1	1	−1	−1	−1	−1
R_3	1	−1	1	−1	1	−1	1	−1
R_4	1	−1	1	−1	−1	1	−1	1

Let $M_\rho = \pm i$, $M_\kappa = \pm 1$

$\Rightarrow M_{\kappa\rho} = \pm i \Rightarrow M^2_{\kappa\rho} = -1$

But $(\kappa\rho)^2 = e \Rightarrow M^2_{\kappa\rho} = +1$

⊙ $M_\rho = \pm i$ does not lead to ~~any~~ valid irrep.

Orthogonality

Def $\chi(M) = \sum_{i=1}^{n} M_{ij}$

Thm $\sum_{p=1}^{n} \overline{\chi^p(CS_i)}\,\chi^p(CS_j)\,N_i = |\Gamma|\delta_{ij}$

$n \equiv$ no. of inequivalent irreps

$p \equiv$ rep. of Γ

$N_i \equiv$ no. of elements in CS_i

$CS_i \equiv$ class

$\sum_{i=1}^{m} \overline{\chi^p(CS_i)}\,\chi^q(CS_i)\,N_i = |\Gamma|\delta_{pq}$

e.g.) $\Gamma = D_4$

$\sum_{p=1}^{5} \overline{\chi^p(CS_i)}\,\chi^p(CS_j)\,N_i = 8\delta_{ij}$

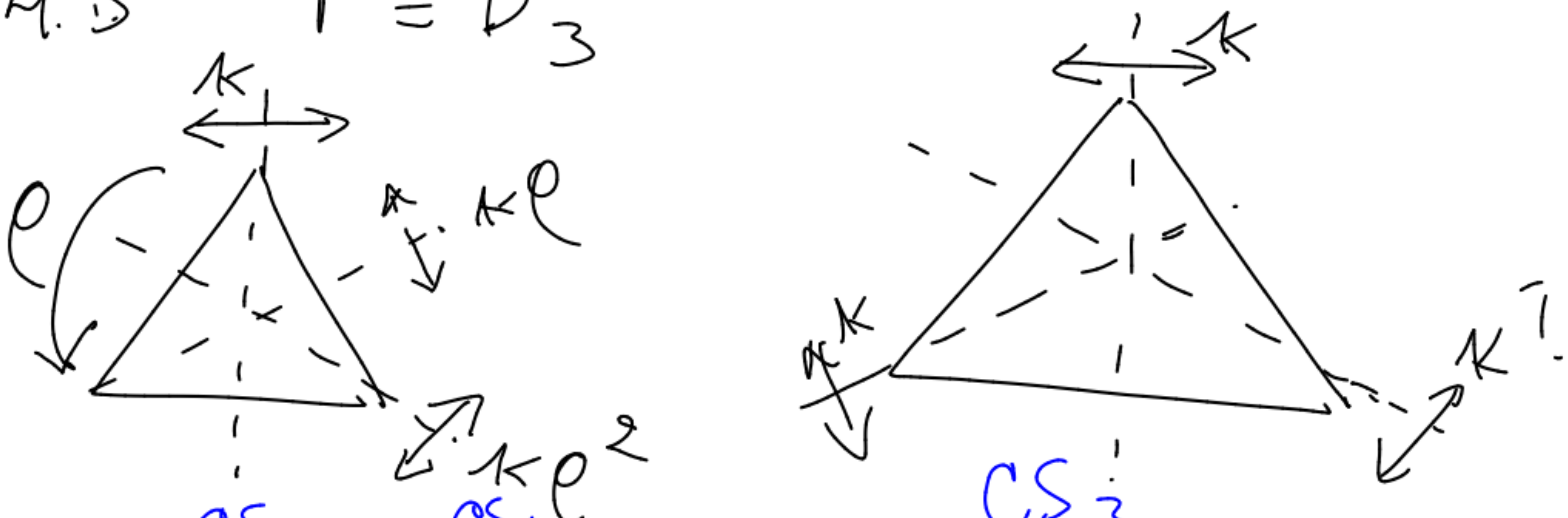
Let $i=1$, $j=5$

$CS_1 = \{e\}$, $CS_5 = \{e^2\}$

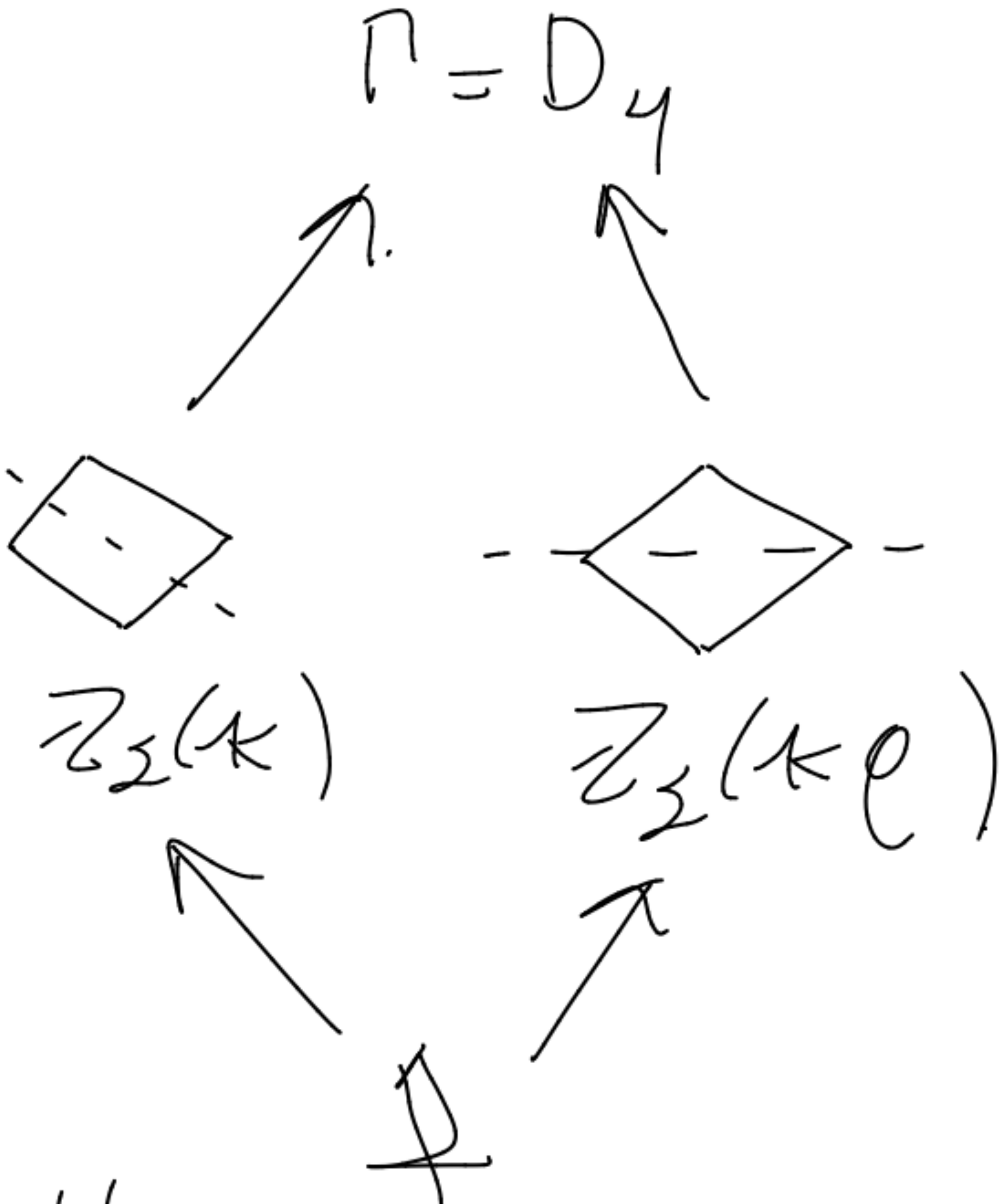
$\overline{\chi^p(\{e\})}\,\chi^p(\{e^2\})\times) = 8\times 0$

$$
\begin{array}{c|ccccc}
 & \{e\} & \{\kappa, \kappa\rho^2\} & \{\kappa\rho, \kappa\rho^3\} & \{\rho, \rho^3\} & \{\rho^2\} \\
\hline
R_1 & 1 & 1 & 1 & 1 & 1 \\
R_2 & 1 & 1 & -1 & -1 & 1 \\
R_3 & 1 & -1 & -1 & 1 & 1 \\
R_4 & 1 & -1 & 1 & -1 & 1 \\
R_5 & 2 & 0 & 0 & 0 & -2
\end{array}
\quad \Bigg\} \quad M_{\gamma_1\gamma_2} = M_{\gamma_1} M_{\gamma_2}
$$

4.3 $\Gamma = D_3$



	CS_1	CS_2		CS_3			
D_3	e	ρ	ρ^2	κ	$\kappa\rho$	$\kappa\rho^2$	
R_1	1	1	1	1	1	1	Trivial
R_2	1	1	1	-1	-1	-1	Nontrivial
R_3	2	-1	-1	0	0	0	Natural



Lattice of Isotropy Subgroups