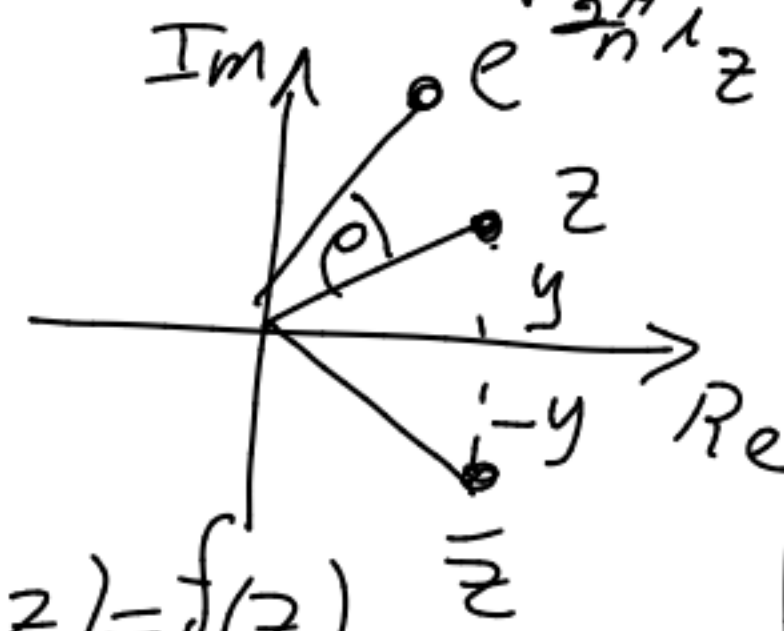
Let $\Gamma = D_n$ acting on $\mathbb{C} = \mathbb{R}^2$

$$\rho \cdot z = e^{\frac{2\pi}{n} i} z$$



$$\kappa \cdot z = \bar{z} \qquad f(\gamma z) = f(z)$$

<u>Claim 1</u> For every D_n-inv polynomial $f(z)$ $\exists\, g : \mathbb{R}^2 \to \mathbb{R}$ s.t.

$$f(z) = g(z\bar{z},\ z^n + \bar{z}^n)$$

<u>Pf</u> Hilbert basis:

$$f(z) = \sum \underbrace{a_{\alpha\beta}}_{\text{coeff.}} \underbrace{z^\alpha \bar{z}^\beta}_{\text{basis monomials}}$$

(i) Reality Cond: $f(z) = \overline{f(z)}$

$$\overline{f}(z) = \sum \bar{a}_{\alpha\beta} \bar{z}^\alpha z^\beta$$

$$f(z) \overset{set}{=} \sum a_{\alpha\beta} z^\alpha \bar{z}^\beta$$

$$\Longrightarrow \boxed{a_{\alpha\beta} = \bar{a}_{\beta\alpha}}$$

(ii) Consider κ

$$f(\kappa z) = f(\bar{z}) = \sum a_{\alpha\beta} \bar{z}^\alpha z^\beta$$

$$f(z) \overset{set}{=} \sum a_{\alpha\beta} z^\alpha \bar{z}^\beta$$

$$\Longrightarrow \boxed{a_{\alpha\beta} = a_{\beta\alpha}}$$

Consider ρ

$$f(\rho z) = f\left(e^{\frac{2\pi}{n} i} z\right) = \sum a_{\alpha\beta}\, e^{\frac{2\pi}{n} i(\alpha-\beta)} z^{\alpha} \bar{z}^{\beta}$$

$\underline{\underline{\text{set}}}$

$$\sum a_{\alpha\beta}\, z^{\alpha} \bar{z}^{\beta}$$

$\Rightarrow$ either $a_{\alpha\beta} = 0$ or $e^{\frac{2\pi}{n} i(\alpha-\beta)} = 1$

$$\Downarrow$$

$$\alpha = \beta \mod n$$

e.g.) $\alpha = \beta = 1 \Rightarrow z\bar{z}$

$\alpha = n,\ \beta = 0 \Rightarrow z^n \xrightarrow{a_{\alpha\beta} = a_{\beta\alpha}} z^n + \bar{z}^n$

<u>Claim 2</u> $u_1(z) = z\bar{z},\ u_2(z) = z^n + \bar{z}^n$

<u>Pf</u> $f(z) = \sum a_{\alpha\beta} z^{\alpha} \bar{z}^{\beta}$

$$= \sum A_{\alpha}(z\bar{z})(z^{\alpha} + \bar{z}^{\alpha}),\ \alpha = 0 \mod n$$

$\Rightarrow z\bar{z},\ z^n + \bar{z}^n,\ z^{2n} + \bar{z}^{2n}, \ldots$

<u>Induction</u>

$$z^{(k+1)n} + \bar{z}^{(k+1)n} = (z^n + \bar{z}^n)(z^{kn} + \bar{z}^{kn}) - (z^n \bar{z}^{kn} + \bar{z}^n z^{kn}) =$$

$$(z^n + \bar{z}^n)(z^{kn} + \bar{z}^{kn}) - (z\bar{z})^n \left(\bar{z}^{(k-1)n} + z^{(k-1)n}\right) \checkmark$$

$\Rightarrow$ invoke induction on $k+1, k-$

Claim 3 Every $\underbrace{D_n\text{-equiv.}}_{g(\gamma z)=\gamma g(z)}$ map has the form:

$$g(z)=p(u_1,u_2)z+q(u_1,u_2)\bar z^{n-1}$$

where p and q are D_n-inv., $u_1=z\bar z$, $u_2=z^n+\bar z^n$.

Pf $g(z)=\sum a_{\alpha\beta} z^\alpha \bar z^\beta$ (no reality cond)

k

$$kg(z)=g(kz) \Rightarrow g(\bar z)=\overline{g(z)} \text{ or } \overline{g(\bar z)}=g(z)$$

$$\sum \bar a_{\alpha\beta} z^\alpha \bar z^\beta = \overline{g(\bar z)} \Rightarrow a_{\alpha\beta}=\bar a_{\alpha\beta}$$

$$\Rightarrow a_{\alpha\beta}\in\mathbb{R}$$

ρ

$$\rho g(z)=g(\rho z)\Rightarrow \rho^{-1}g(\rho z)=g(z)$$

$$e^{-\frac{2\pi}{n}i}\, g\!\left(e^{\frac{2\pi}{n}i} z\right) = g(z)$$

$$\| $$

$$\sum a_{\alpha\beta}\, e^{\frac{2\pi}{n}i(\alpha-\beta-1)} z^\alpha \bar z^\beta \overset{\text{want}}{=} \sum a_{\alpha\beta} z^\alpha \bar z^\beta$$

$$\Rightarrow a_{\alpha\beta}=0 \text{ OR } \alpha=\beta+1 \bmod n$$

mod $z\bar z$: $\sum \Big[\underbrace{A_\gamma(z\bar z) z^\gamma}_{z,\, z^{n+1},\, z^{2n+1},\, \dots} + \underbrace{B_\gamma(z\bar z)\bar z^\gamma}_{\bar z^{n-1},\, \bar z^{2n-1},\, \dots}$

$\ell = \beta+1 \mod n \Rightarrow z^{\ell n+1}$ & $\bar z^{(\ell+1)(n-1)}$

$\ell = 0, 1, 2, \ldots$

then

$$z^{(\ell+2)n+1} = (z^n+\bar z^n)\, z^{(\ell+1)n+1} - (z\bar z)^n z^{\ell n+1}$$

$$\bar z^{(\ell+3)n-1} = (z^n+\bar z^n)\, \bar z^{(\ell+2)n-1} - (z\bar z)^n \bar z^{(\ell+1)n-1}$$

$$z^{kn+1} = (z^{kn}+\bar z^{kn})\, z - (z\bar z)\bar z^{kn-1}$$ redundant

$$\bar z^{kn-1} = \underbrace{(z^{(k-1)n}+\bar z^{(k-1)n})}_{\text{inv}}\, \bar z^{n-1} - \underbrace{(z\bar z)}_{\text{inv.}}{}^{n-1} z^{(k-2)n+1}$$

induction $\Longrightarrow$ $z, \bar z^{n-1}$ suffice

Thm Every D_n-equiv. polynomial has the form:

$$\boxed{g(z) = p(z\bar z, z^n+\bar z^n)\, z + q(z\bar z, z^n+\bar z^n)\, \bar z^{n-1}}$$

D_n-inv

$$\boxed{\frac{dz}{dt} = g(z)}$$

e.g.) 4.3, $\Gamma = D_3$

$$\boxed{g(z) = p(z\bar z, z^3+\bar z^3)\, z + q(z\bar z, z^3+\bar z^3)\, \bar z^2}$$

$$= (p_0 z\bar z + p_1 (z\bar z)^2 + \dots + p_2 z^3 + \bar z^3 + \dots)\, z$$