Hopf Bif. with Symmetry

Let $\Gamma = O(2)$ on $\mathbb{R}^2 = \mathbb{C}$

$$\theta \cdot z = e^{\theta i} z, \quad \theta \in SO(2)$$

$$\kappa \cdot z = \bar{z}$$

Recall: The natural action of $O(2)$ on $\mathbb{R}^2$ is abs. irred.

$\Rightarrow$ Commuting matrices are of the form

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \quad a \in \mathbb{R}$$

$\Rightarrow$ Hopf bif. cannot occur.

Consider $\Gamma = O(2)$ on $\mathbb{R}^4 = \mathbb{C}^2$

$$\theta(z_1, z_2) = (e^{\theta i} z_1, e^{\theta i} z_2)$$

$$\kappa(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$$

Commuting matrices are:

$$\begin{bmatrix} aI_2 & bI_2 \\ cI_2 & dI_2 \end{bmatrix}$$

e-vals are those of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ repeated twice

Let $a = d = 0$, $b = -1$, $c = 1$

e-vals: $0 \pm i$ (twice)

$\Rightarrow$ Hopf bif. is possible ✓

e.g.) $\Gamma = SO(2)$

Natural rep on $\mathbb{R}^2 = \mathbb{C}$ is

$$\theta \cdot z = e^{\theta i} z$$

① This rep. is non-abs. irred.

$\theta = \frac{\pi}{2} \in SO(2) \Rightarrow M_\theta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, e-vals $0 \pm i$

$M_\theta SO(2) = SO(2) M_\theta$ since $SO(2)$ is Abelian

$\Rightarrow$ Hopf. bif may occur for a 2×2 sys. of ODEs with $SO(2)$ symmetry

Recall: isotypic components

$$V = W_1 \oplus W_2 \oplus \dots \oplus W_s$$

Let $A : V \xrightarrow{\text{linear}} V$

Then $A(W_k) \subset W_k$

think of as (df)

<u>Lemma</u> Let $L : \mathbb{R}^n \to \mathbb{R}^n$ linear with a nonreal e-val and commuting with Γ.

$$\mathbb{R}^n = V_1 \oplus \dots \oplus V_k \quad (*)$$

Then

(i) Some abs. irrep. of Γ occur at least twice in $(*)$

(ii) The action of Γ on some V_j's is not abs. irred.

Consider $\frac{dx}{dt} = f(x, \mu)$, $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}$

st. $\gamma f(x,\mu) = f(\gamma x, \mu)$, $\forall \gamma \in \Gamma$

$$f(0,\mu) = 0 \qquad \forall \mu$$

Need to find cond. on the action of Γ that will lead to purely imag. vals in $(df)_{(0,0)}$

Claim There must be Γ-inv. subspace of $\mathbb{R}^n$ that is either

$W = V \oplus V$, V is abs. irred.

W is irred. but not abs. irred.

In either case, at the bif. point we get:

$$(df)_{(0,0)} = \begin{bmatrix} 0 & I_p \\ -I_p & 0 \end{bmatrix}, \quad p = \frac{n}{2}$$

e-vals: $\sigma(\mu) \pm \omega(\mu) i$ of multip. p

Def. A rep. W of Γ is Γ-simple either if it is composed of two copies of an abs. irrep. V st $W = V \oplus V$ or if W is irred. but not abs. irred.

Spatio-Temporal Symmetries

Def Let $x(t)$ be a per. sol.

Then γ is a spatial symmetry if $\gamma x(t) = x(t), \ \forall t$

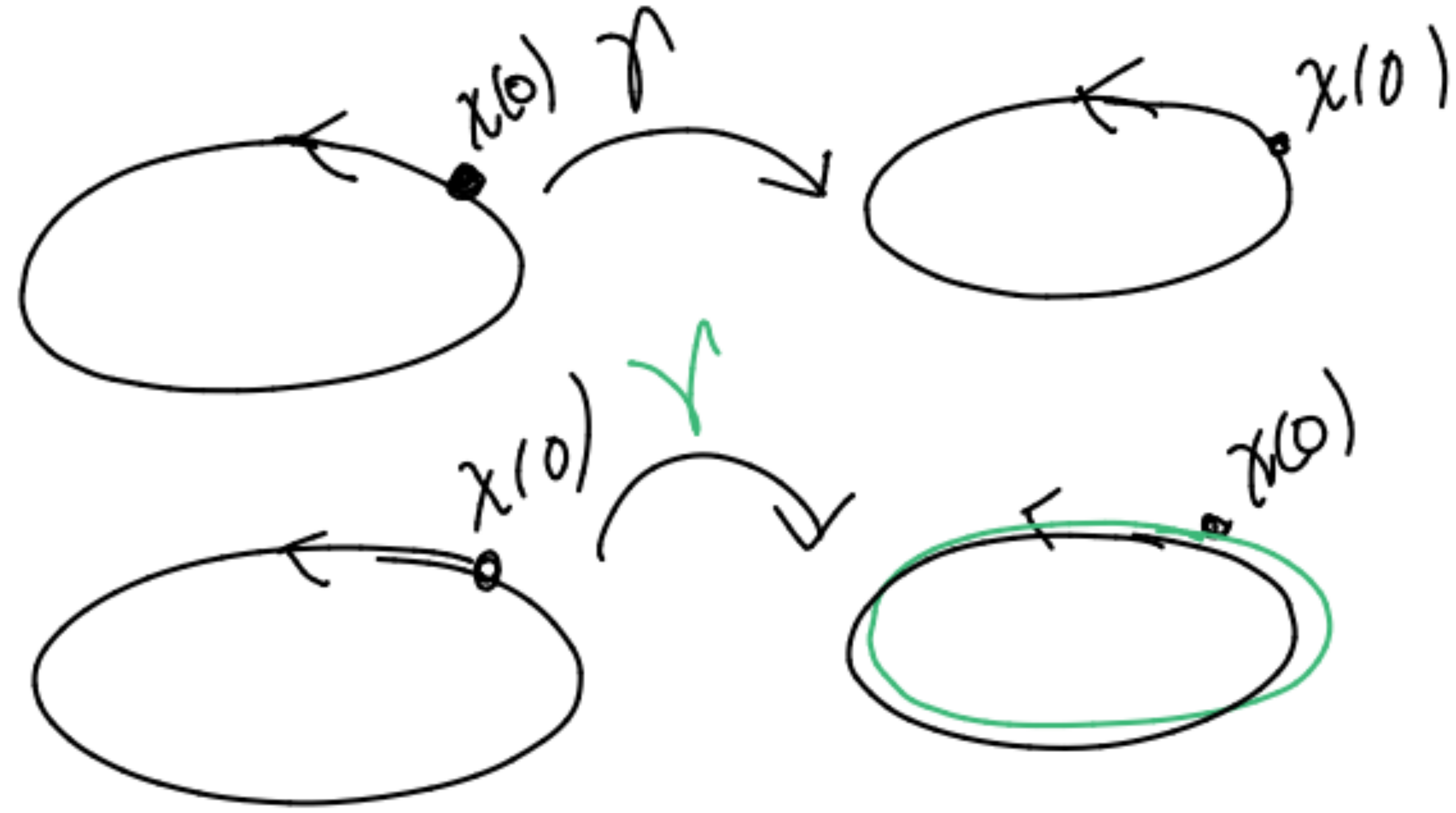
e.g.) Flames

Assume per. $T = 2\pi$

Then (γ, θ) is a spatio-temporal symmetry if

$$(\gamma,\theta)x(t) = \gamma(t+\theta) = x(t), \ \forall t$$

$$\gamma\{x(t)\} = \{x(t)\}$$ ← trajectory



Def Isotropy Subgroup

$$\Sigma_{x(t)} = \{(\gamma,\theta) \in \Gamma \times S^1 : \gamma x(t+\theta) = x(t)\}$$

Action of S^1

Consider: $\frac{dx}{dt} = Jx$ ($J \equiv$ jacobian)

Sol: $x(t) = e^{tJ} x(0)$

$$(e,\theta)\cdot x(t) = x(t+\theta) = e^{(t+\theta)J}x(0) = e^{\theta J}e^{tJ}x(0) = e^{\theta J}x(t)$$

$$e^{\theta J} \overset{\text{Taylor}}{=} \begin{bmatrix} I_p & 0 \\ 0 & I_p \end{bmatrix} + \begin{bmatrix} 0 & \theta I_p \\ -\theta I_p & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -\theta^2 I_p & 0 \\ 0 & -\theta^2 I_p \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 0 & -\theta^3 I_p \\ \theta^3 I_p & 0 \end{bmatrix} + \cdots = \begin{bmatrix} \cos\theta\, I_p & \sin\theta\, I_p \\ -\sin\theta\, I_p & \cos\theta\, I_p \end{bmatrix}$$

$$\Rightarrow \boxed{\theta\cdot x(t) = e^{\theta J}x(t)}$$

S' commutes with the action of Γ

because γ acts on range and $\theta\in S'$ acts on the domain of f $\Rightarrow \theta\cdot(V_1,V_2) = \underbrace{\begin{bmatrix} \cos\theta\, I & \sin\theta\, I \\ -\sin\theta\, I & \cos\theta\, I \end{bmatrix}}_{R_\theta}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

$\Rightarrow \gamma\cdot\theta = \theta\cdot\gamma$

Consider $W = V\oplus V$

he action of $\Gamma \times S^1$ on $V \oplus V$
s
$$\gamma(V_1, V_2) = (\gamma V_1, \gamma V_2), \ \forall \gamma \in \Gamma$$
$$\theta(V_1, V_2) = R_\theta (V_1, V_2)$$

m (Equivariant Hopf Lemma)
$$\frac{dx}{dt} = f(x, \mu), \ x \in \mathbb{R}^n, \ \mu \in \mathbb{R}$$
a Γ-equiv. Hopf bif. problem
$$\gamma f(x, \mu) = f(\gamma x, \mu), \ \forall \gamma \in \Gamma$$
$$(df)_{(0,0)} = J$$
$$\lambda_\pm = \sigma(\mu) \pm w(\mu) i$$

If Γ acts Γ-simply on $\mathbb{R}^n$ and $\Sigma \subset \Gamma \times S^1$ is an isotropy subgroup st.
$$\dim(\operatorname{Fix} \Sigma) = 2$$
then $\exists$ a unique branch of per. sol. $x(t)$ with period close to ~~2π~~ bifurcating from $(0,0)$ st. $\Sigma_{x(t)}$ is the isotropy subgroup.

<u>Example</u>: Hopf Bif. with $O(2)$ symmetry

Let $\Gamma = O(2)$ on $\mathbb{R}^4 = \mathbb{C}^2$

$$\boxed{\begin{aligned} \rho(w_1, w_2) &= (e^{\rho i} w_1, e^{\rho i} w_2) \\ \kappa(w_1, w_2) &= (\bar{w}_1, \bar{w}_2) \end{aligned}}$$

Change of coord:

$$z_1 = \frac{1}{2}(\bar{w}_1 - \bar{w}_2 i)$$

$$z_2 = \frac{1}{2}(w_1 - w_2 i)$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ -i \end{bmatrix} \right\}$$

$$\mathbb{C}^2 = V_1 \oplus V_2$$

$$\rho(z_1, z_2) = (e^{-\rho i} z_1, e^{\rho i} z_2)$$

$$\kappa(z_1, z_2) = (z_2, z_1)$$

$$\theta(z_1, z_2) = (e^{\theta i} z_1, e^{\theta i} z_2)$$

extra symmetry (Birkhoff Normal Forms)

What are the possible Hopf Bif. that can occur?

Sol	Orbit	Isotropy Subgroup Σ	Fixed Point Subspace	dim Fix(Σ)
Trivial	$(0,0)$	$O(2)\times S^1$	$\{0\}$	0
Rotating Wave	$(a,0), a>0$	$\widetilde{SO(2)}$	$\{(z_1,0)\}$	2
Standing Wave	$(a,a), a>0$	$Z_2 \oplus Z_2^c$	$\{(z_1,z_1)\}$	2
General Points	$(a,b), a>b>0$	Z_2^c	$\{(z_1,z_2)\}$	4