Recall: $\Gamma = O(2)$ on $\mathbb{C}^2$:

$$\rho(z_1, z_2) = (e^{-\rho i} z_1, e^{\rho i} z_2)$$

$$\kappa(z_1, z_2) = (z_2, z_1)$$

$$\theta(z_1, z_2) = (e^{\theta i} z_1, e^{\theta i} z_2)$$

What are the possible Hopf bif that can occur?

Irreps	Interpretation
$\mathbb{R}$ Trivial	Every per sol is fixed $\forall t$ by $\gamma \in O(2)$, $\forall \gamma$
$\mathbb{R}$ Nontrivial	$SO(2)$ acts trivially κ acts by multip. by -1 (γ, π) symmetry, $\forall \gamma \in SO(2)$

$\mathbb{C}$, $z \mapsto e^{k\varphi i} z$	Every $\mathbb{C}$-axial subgroup contains $\mathbb{Z}_k = \langle \frac{2\pi}{k} \rangle \subset O(2)$ $\frac{2\pi}{k} \cdot v(s) = v(s)$

$$\Gamma = O(2)$$
$$|$$
$$\mathbb{Z}_k$$

Isotropy Lattice

Trivial Symmetries

$(0,0)$ — identity

(π,π) — R_π with $\frac{1}{2}$ period shift

phase

$$\mathbb{Z}_2^c = \{(0,0), (\pi,\pi)\}$$

$\mathbb{Z}_2^c$ fixes $\mathbb{C}^2 \simeq \mathbb{R}^4$ spanned by (z_1, z_2)

I. $\Sigma_1 = \widetilde{SO(2)} = \{(\underset{\in SO(2)}{\theta}, \underset{\theta \in S^1}{\theta}) : \theta \in S^1\}$

$$(\theta,-\theta)\cdot(z_1,z_2) = (e^{2\theta i} z_1, z_2)$$

$$(\theta, id)\cdot(z_1,z_2) = (e^{-\theta i} z_1, e^{\theta i} z_2)$$

$$(id, \theta)\cdot(e^{-\theta i} z_1, e^{\theta i} z_2) = (z_1, e^{2\theta i} z_2)$$

$$\mathrm{Fix}(\Sigma_1) = \{(0, z_2)\}$$

$$(\theta,\theta)\cdot(z_1,z_2) = (z_1, e^{2\theta i} z_2)$$

$$\Rightarrow \mathrm{Fix}(\Sigma_1) = (z_1, 0)$$

$(\theta,\theta) \equiv$ Rotation by θ is the same as phase shift in time by $-\theta$

Pattern

$u(\theta) = R_\theta u(0)$ — Rigid or uniform rotation

II. $\Sigma_2 = \mathbb{Z}_2(\kappa) = \{\kappa\}$

$$\mathrm{Fix}(\Sigma_2) = \{(z,z) : z \in \mathbb{C}\}$$

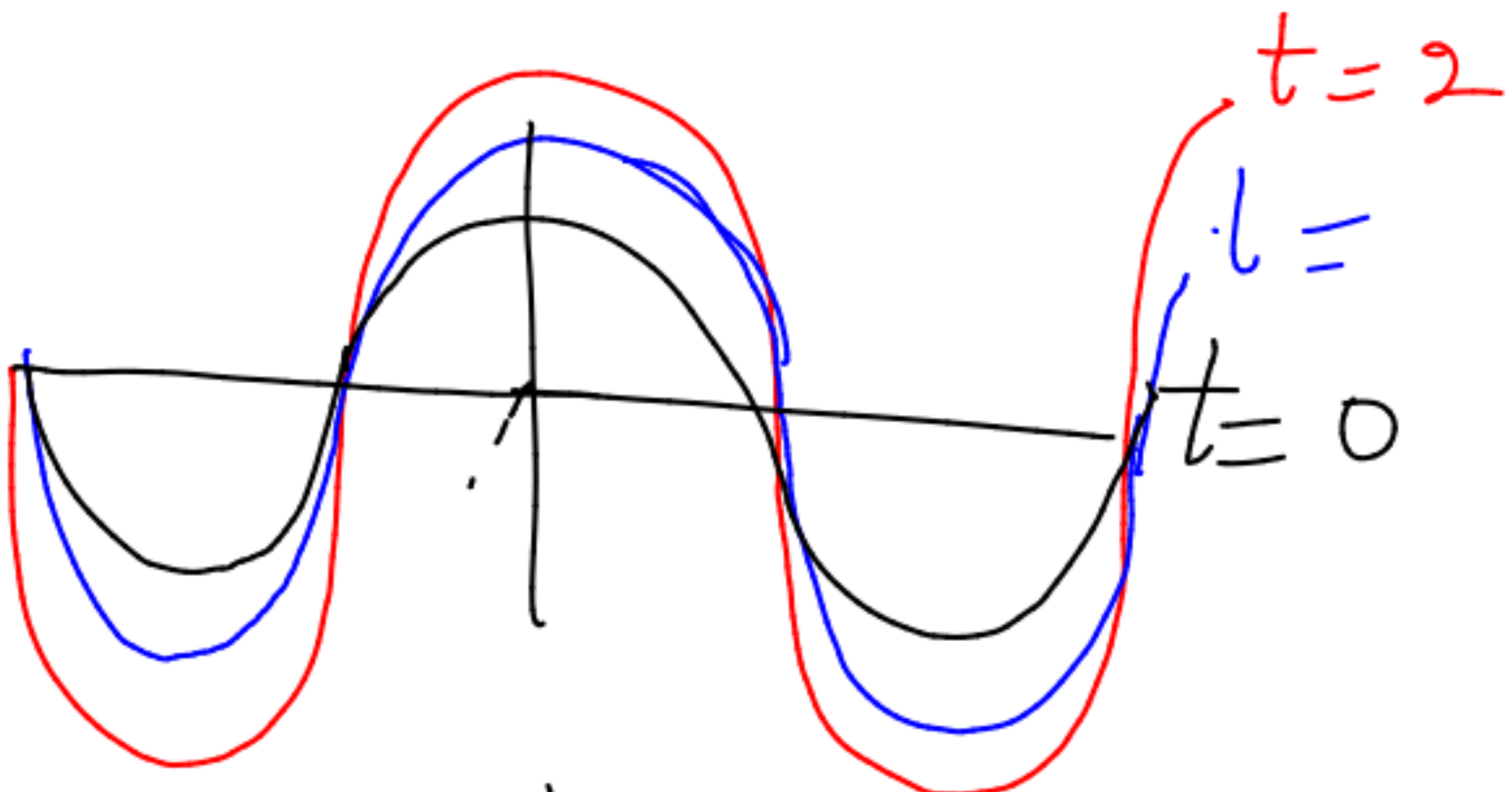
⊙ Axis of reflection is 2D $z \in \mathbb{R}$ and does not vary, so the sol. cannot rotate, only oscillate in place.

$\Rightarrow \exists$ a Standing Wave: $\kappa u(s) = u(s)$

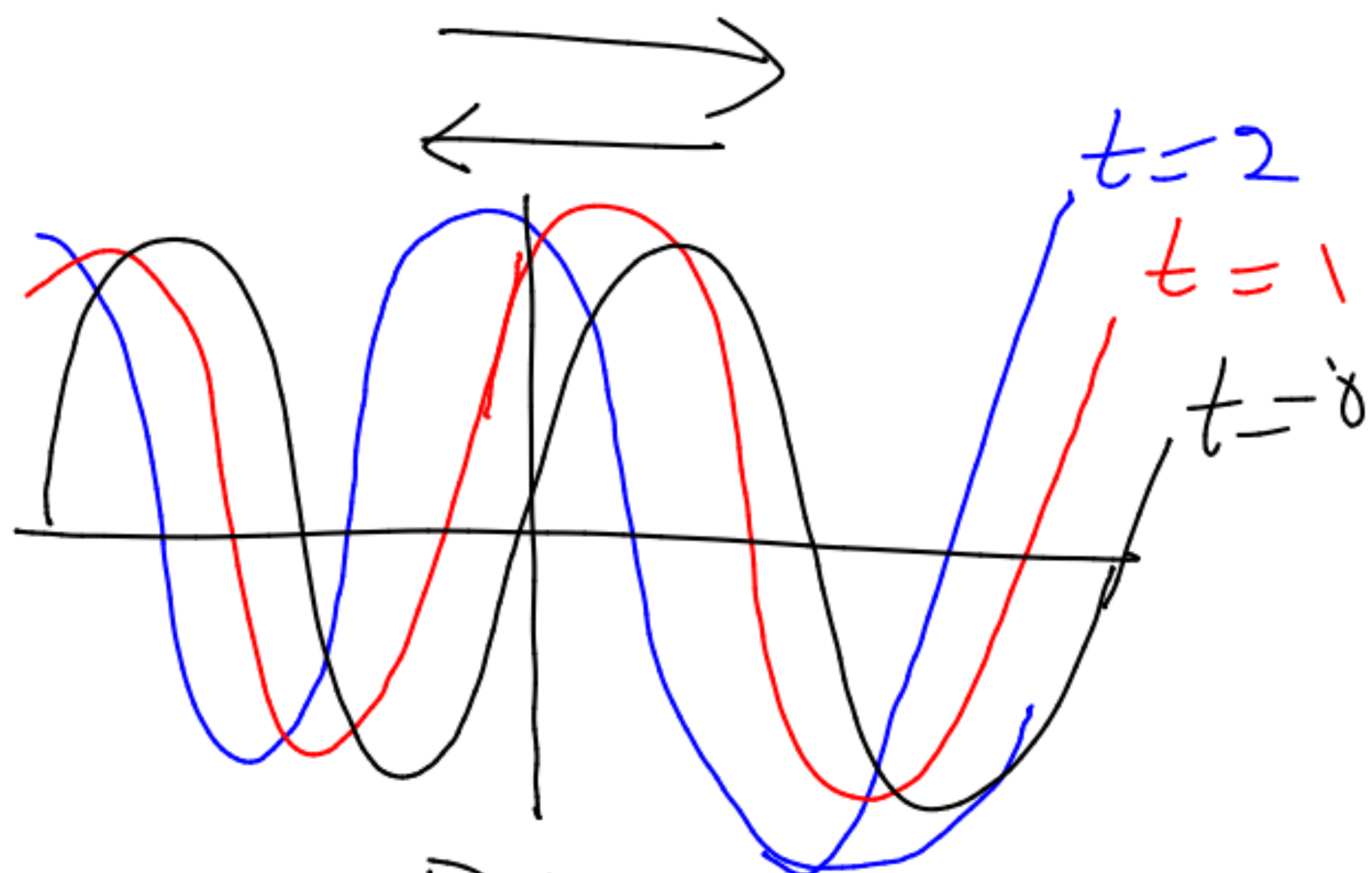
$$\Sigma_2 = \mathbb{Z}_2 \times \mathbb{Z}_2^c = \{(0,0),(\pi,\pi),(k,0),(k,0),(\pi,\pi)\}$$

<u>Up to conjugacy</u>, Σ_1 and Σ_2 are the only $\mathbb{C}$-axial subgroups of $O(2)\times S^1$ on $\mathbb{C}^2$

<u>Visualization:</u>



SW



RW

(z_1, z_2) is conjugate to (x_1, x_2)

$x_1, x_2 \geq 0$

By k-symmetry can assume

$x_2 \geq x_1 \geq 0$

$x_1 = x_2 = 0$ $\quad \Sigma = \Gamma \times S^1$

$x_2 > x_1 = 0$ $\quad \Sigma_x = \Sigma_{RW}$

$x_2 = x_1 > 0$ $\quad \Sigma_x = \Sigma_{SW}$

$x_2 > x_1 > 0$ $\quad \Sigma_x = \{(\pi, \pi)\}$

$\mathrm{Fix}(\{(\pi, \pi)\}) = \mathbb{C}^2$

Branch	Orbit Representative	Isotropy Σ	Fixed Pt. Subspac	dim Fix(Σ)
Trivial	$(0,0)$	$O(2)\times S^1$	$\{0\}$	0
$\Sigma_1 \equiv$ Rotating Wave	$(a,0)$, $a>0$	$\widetilde{SO(2)}$	$\{(z_1,0)\}$	2
$\Sigma_2 \equiv$ Standing Wave	(a,a), $a>0$	$Z_2\times Z_2^c$	$\{(z,z)\}$	2
General Points	(a,b), $a>b>0$	Z_2^c	$\{(z_1,z_2)\}$	4

$$
\begin{tikzcd}
& O(2)\times S^1 & \\
\widetilde{SO(2)} \arrow[ur] & & \mathbb{Z}_2\times\mathbb{Z}_2^{c} \arrow[ul] \\
& \mathbb{Z}_2^{c} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Proposition

(i) Every $O(2)\times S^1$ inv. f has the form:

$$f(z_1, z_2) = P(N, \Delta) \quad \underline{\text{Taylor}} \ldots$$

$$N = |z_1|^2 + |z_2|^2$$

$$\Delta = \delta^2, \quad \delta = |z_2|^2 - |z_1|^2.$$

(ii) Every $O(2)\times S^1$ equiv. g has the form:

$$g(z_1, z_2) = (p+qi)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + (r+si)\delta \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

where p, q, r, s are $O(2)\times S^1$ inv.

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = g(z_1, z_2) \overset{\text{set}}{=} 0$$

Assume $z_1 = a \geq 0$, $z_2 = b \geq 0$ are real

Solving $g = 0$ is the same as solving

$(p+qi)a + (r+si)\delta a = 0$

$(p+qi)b - (r+si)\delta b = 0$

Possibilities

(i) $a = b = 0$

(ii) $b = 0$, $\begin{matrix} p + r\delta = 0 \\ q + s\delta = 0 \end{matrix} \Rightarrow \left.\begin{matrix} p - a^2 r = 0 \\ q - a^2 s = 0 \end{matrix}\right\}$ RW

(iii) $\begin{matrix} p + r\delta = 0 \\ q + s\delta = 0 \\ p - r\delta = 0 \\ q - s\delta = 0 \end{matrix} \Longleftrightarrow \begin{matrix} p = q = 0 \\ r\delta = s\delta = 0 \end{matrix}$

(a) $\delta = 0$, $p = q = 0$

$\Rightarrow a = b$, $p = q = 0$

SW

(b) $p = q = r = s = 0$

Orbit	Isotropy	Branching Eq.
$(0,0)$	$O(2)\times S^1$	—
$W(a,0)$	$\widetilde{SO(2)}$	$p-a^2r=0$ $q-a^2s=0$
$W(a,a)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2^c$	$p=0$ $q=0$
(a,b)	$\mathbb{Z}_2^c$	$p=q=r=s=0$

Generically, $r(0)\neq 0$, $s(0)\neq 0$

$\Rightarrow$ no sol. with $\Sigma_{\mathbb{Z}_2^c}$ symmetry

SW

$$p(N,\Delta,\delta) - r(N,\Delta,\delta)\,\delta = 0$$

Set $x_1 = a$, $x_2 = 0$

$$\Rightarrow p(a^2, a^4, \lambda) - a^2 r(a^2, a^4, \lambda) = 0$$

$$[p_N(0) - r(0)]a^2 + p_\lambda(0)\lambda + \dots = 0, \quad p_\tau(0) = 0$$

$$\Longrightarrow \lambda = -\frac{p_N(0) - r(0)}{p_\lambda(0)} a^2$$

CM Reduction

$p(0) = 0$

$q(0) = 0$

$p_\tau(0) = 0$

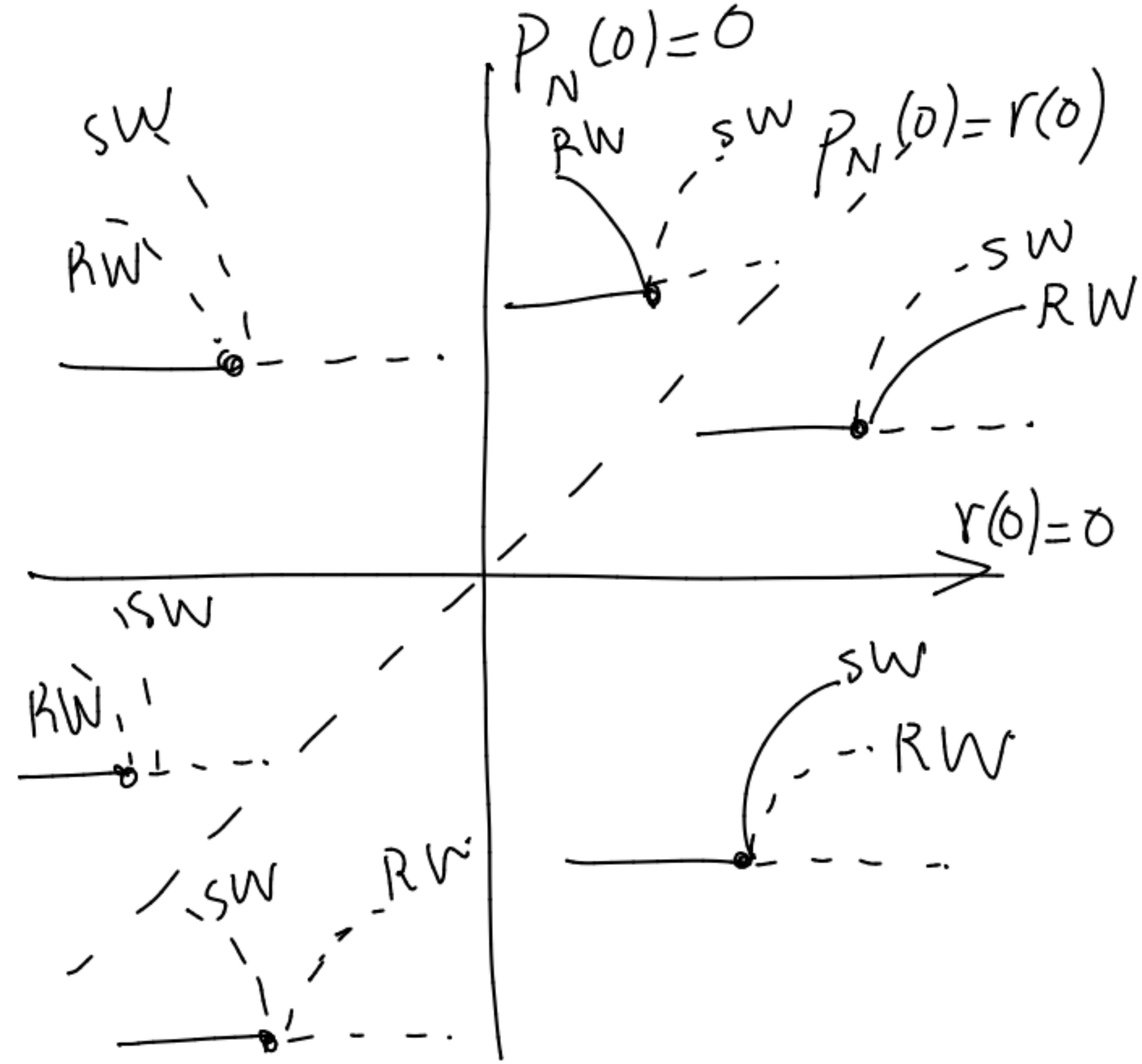
$q_\tau(0) = -1$

$p_\lambda(0) \neq 0$ (e-value crossing cond.)

Stability

Sol.	Isotropy	Branching
Triv	$O(2)\times S^1$	$z=0$
RW	$\widetilde{SO(2)}$	$\lambda = \frac{(-p_N(0)+r(0))}{p_\lambda(0)} a^2 + \ldots$
SW	$Z_2 \oplus Z_2^c$	$\lambda = \frac{-2p_N(0)}{p_\lambda(0)} a^2 + \ldots$

$$a^2 = \sqrt{\frac{\lambda p_\lambda(0)}{r(0)-p_N(0)}}$$



$p_\lambda(0) < 0$