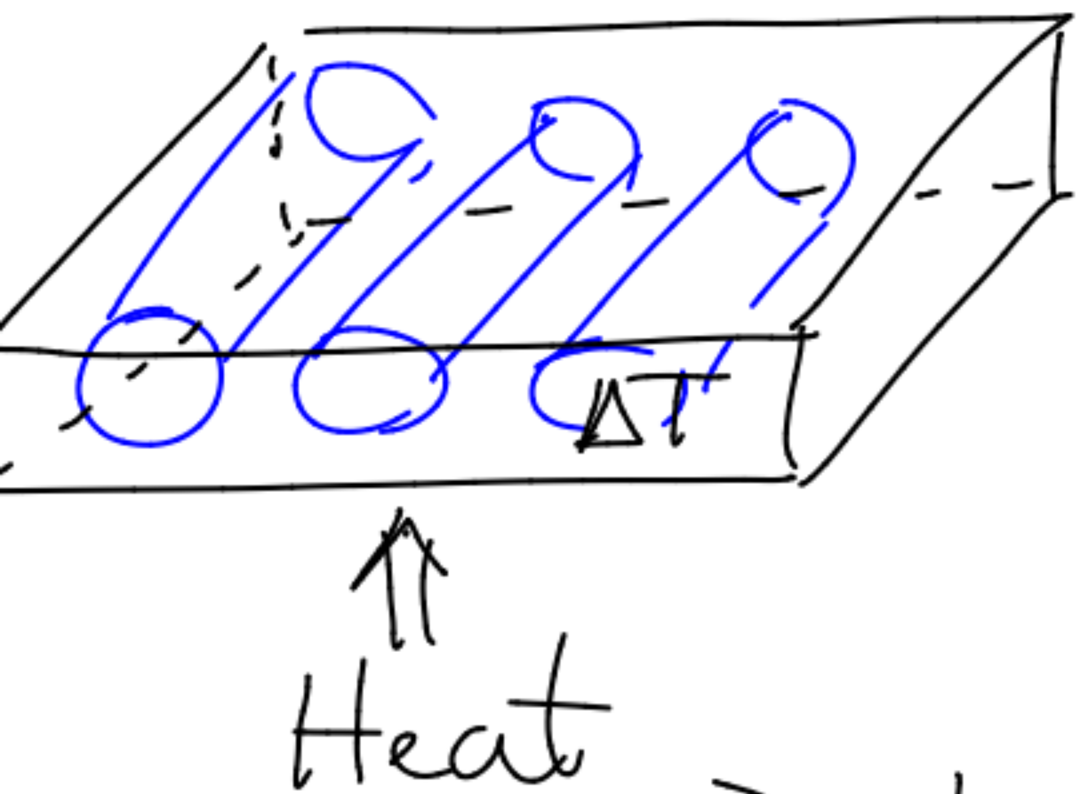
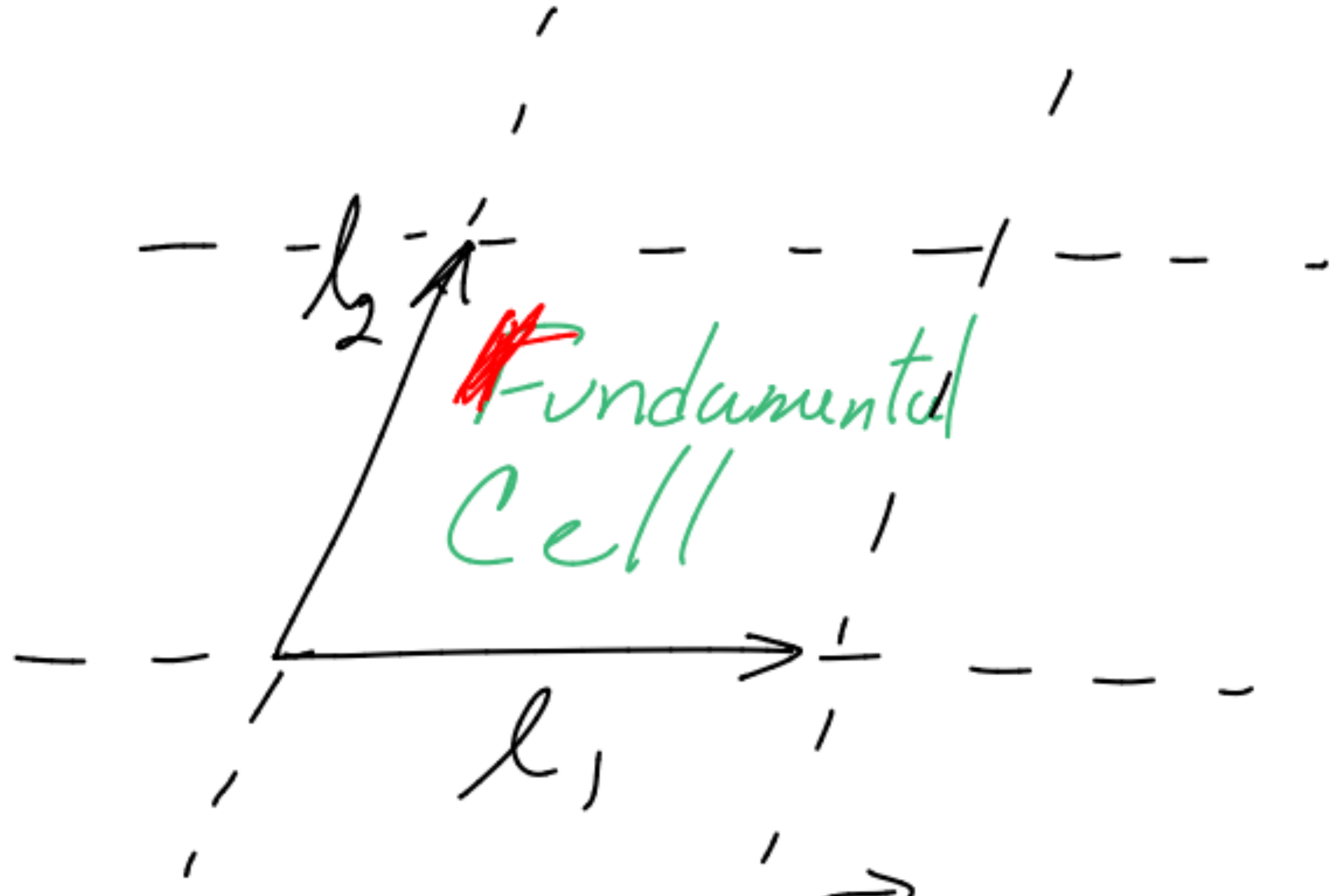
Ch 5 Lattice Patterns



Let $\vec{l}_1$ & $\vec{l}_2$ be two lin. indep. vectors in $\mathbb{R}^2$

Suppose: $u(x,t): \mathbb{R}^2 \to \mathbb{R}^1$

u is doubly periodic w.r.t. $\vec{l}_1$ and $\vec{l}_2$ if $u(x+\vec{l}_1,t) = u(x,t)$
$= u(x+\vec{l}_2,t)$



Let $\vec{l} = n_1\vec{l}_1 + n_2\vec{l}_2$, $n_1, n_2 \in \mathbb{Z}$

$$u(x+\vec{l}, t) \overset{\text{want}}{=} u(x,t)$$

Def. Let $\mathcal{L} = \{\vec{l} = n_1\vec{l}_1 + n_2\vec{l}_2 : n_1, n_2 \in \mathbb{Z}\}$ planar lattice (infinite discrete group of $\mathbb{R}^2$)

Def $C^\infty_{\mathcal{L}} = \left\{ \begin{array}{l} \text{All smooth functions} \\ \text{that are doubly per.} \\ \text{w.r.t. } \mathcal{L} \end{array} \right\}$

$$C^\infty_{\mathcal{L}} = \left\{ u \in C^\infty_{\mathcal{L}} : u(x+\ell, t) = u(x,t), \ \forall \ell \in \mathcal{L} \right\}$$

Remarks

- The lattice $\mathcal{L}$ is a subgroup of translations
- $C^\infty_{\mathcal{L}}$ is the fixed-point subspace of the action of $\mathcal{L}$ on $C^\infty_{\mathcal{L}}$
- There are five kinds of planar lattices: rhombic, square, hexagonal, rectangular and oblique

3. $C^\infty_{\mathcal{L}}$ is an inv. subspace

5. The lattices that are most relevant for bif. theory are those generated by two vectors of the same length: rhombic, square, hexagonal.

Def. $\mathcal{L}^* = \{ n_1 \vec{k}_1 + n_2 \vec{k}_2 : n_1, n_2 \in \mathbb{Z} \}$

$\vec{k}_1, \vec{k}_2 \in \mathbb{R}^2$ s.t.

$$\vec{k}_i \cdot \vec{\ell}_j = 2\pi \delta_{ij}, \quad i,j = 1,2$$

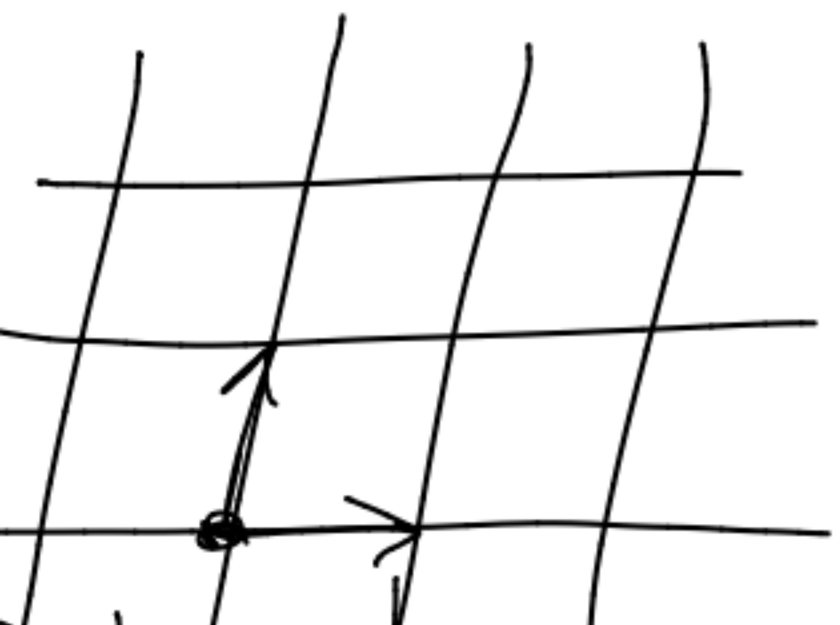
OR

$$\mathcal{L}^* = \{ \vec{k} \in \mathbb{R}^2 : \vec{k} \cdot \vec{\ell} \in \mathbb{Z}\{2\pi\}, \forall \ell \in \mathcal{L} \}$$

Consider $u(x,t)$ as a sum of Fourier modes that lie on the dual lattice:

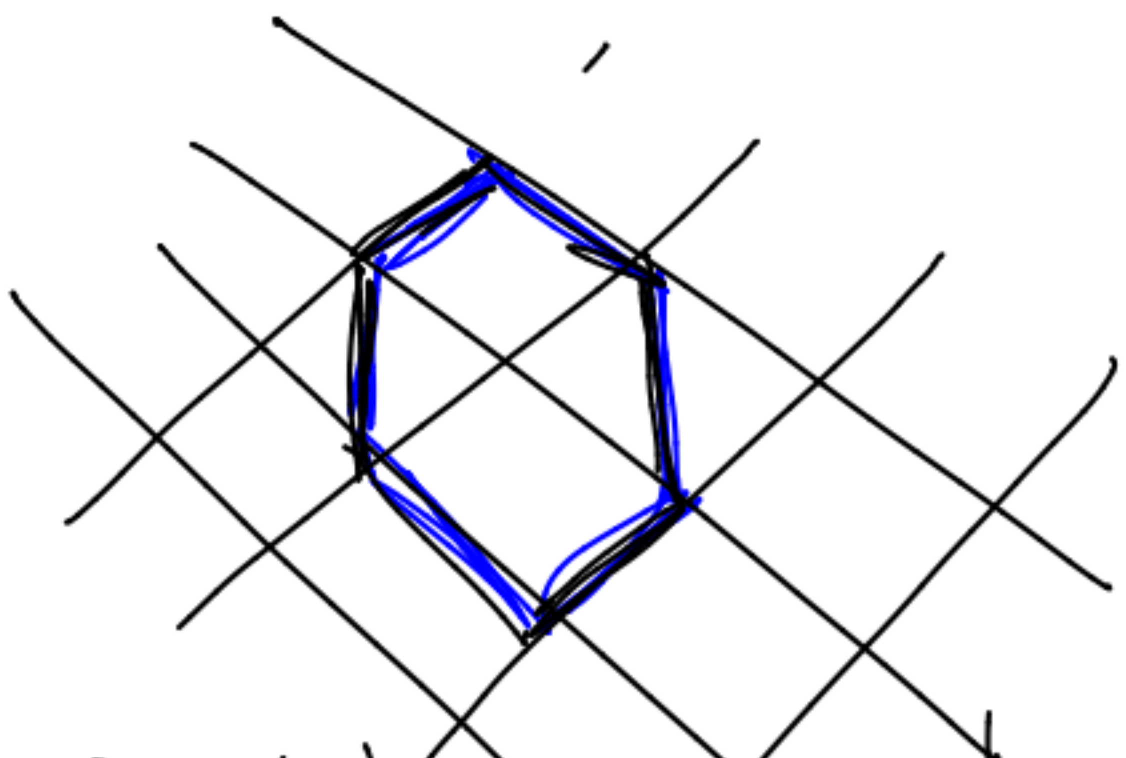
$$u(x,t) = \sum_{k \in \mathcal{L}^*} \underbrace{Z_k(t)}_{\text{amplitude coeff.}} \underbrace{e^{i\vec{k}\cdot\vec{x}}}_{\text{spatial modes}} + c.c.$$

Common dual lattices:

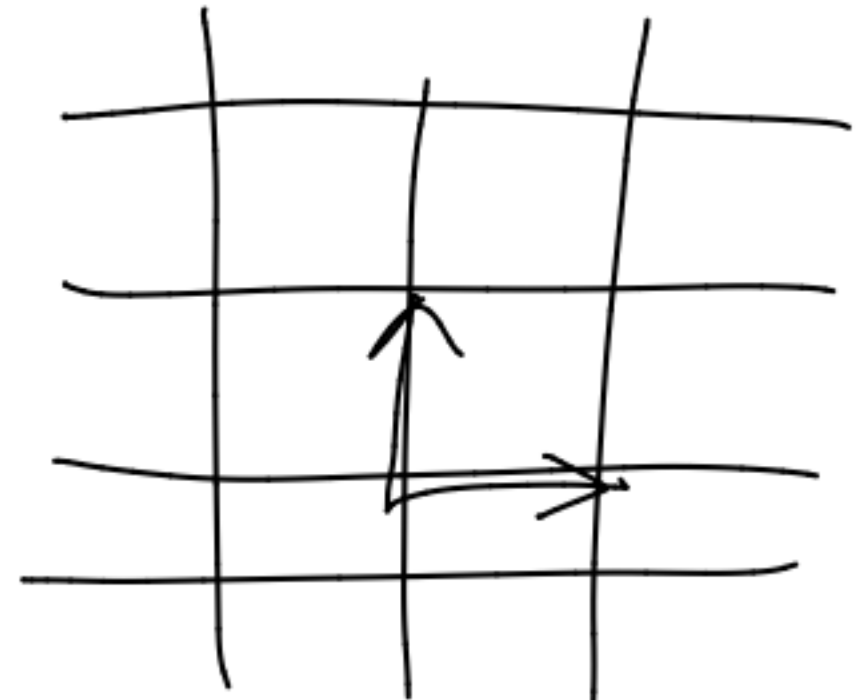


Rhombic

$\vec{k}_1 = k^2\cos\left(\frac{4\pi}{3}\right), \vec{k}_2 = k^2\cos\left(\frac{2\pi}{3}\right)$

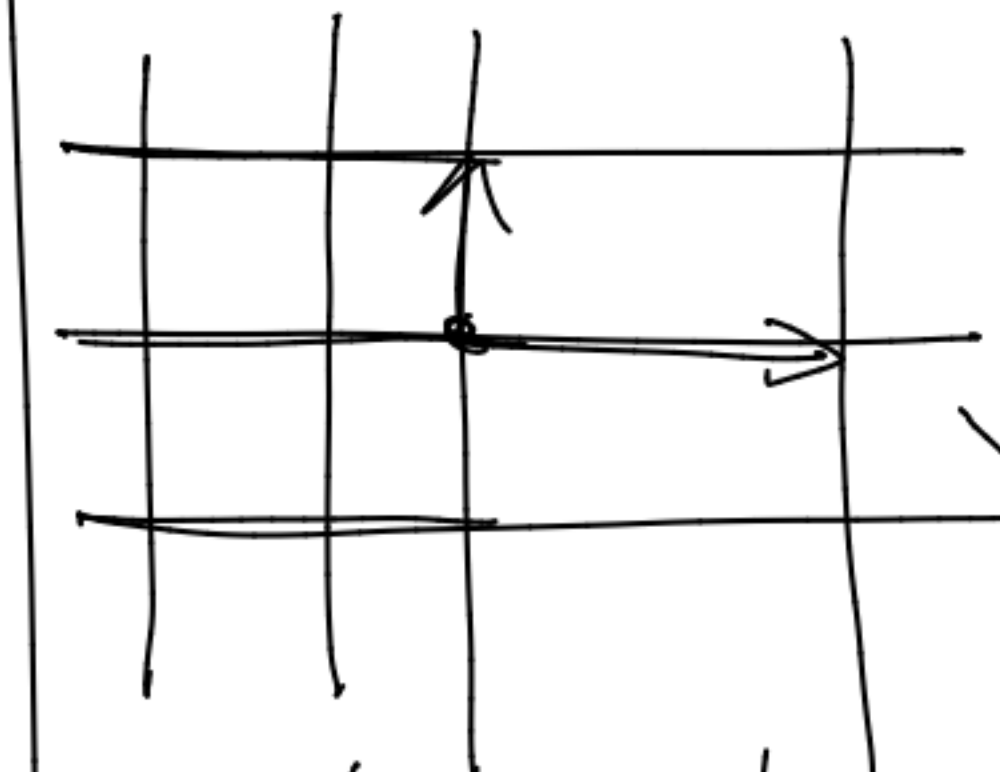


Hexagonal

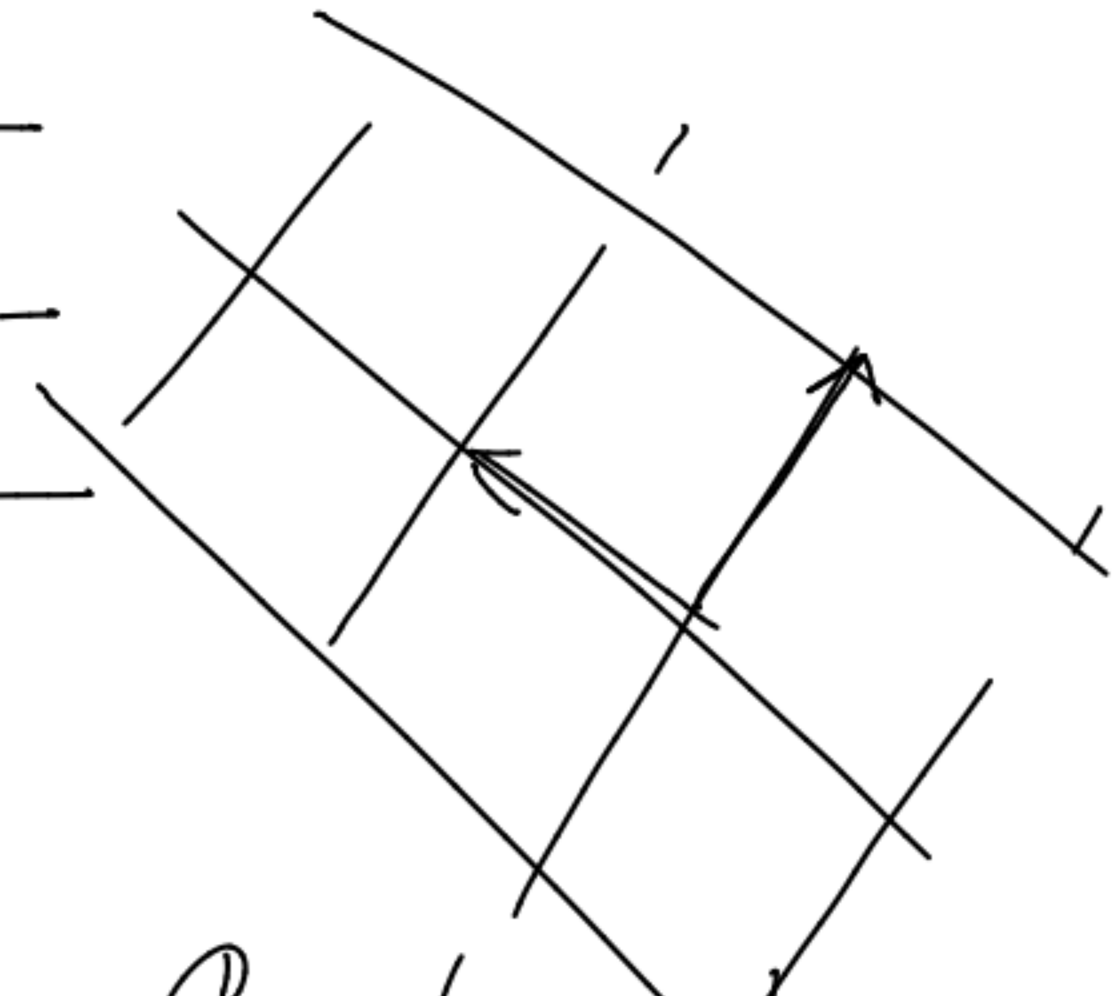


$|\vec{k}_1| = |\vec{k}_2|$

Squares



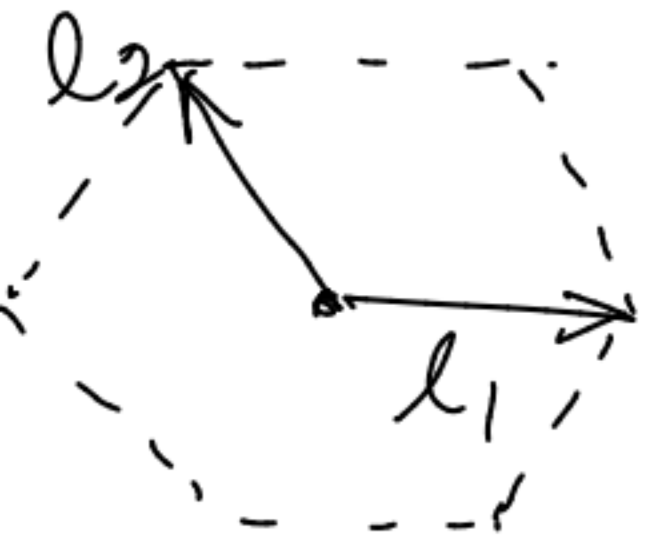
Rectangular



Centered Rectangular

g.) Hexagonal Lattice

$$\mathcal{L} = \left\{ \vec{l}_1 = (c, 0), \; l_2 = \left(-\frac{c}{2}, c\frac{\sqrt{3}}{2}\right) \right\}$$



Let $\vec{k}_1 = (a$

$\vec{k}_1 \cdot \vec{l}_1 = 2\pi$

$\vec{k}_1 \cdot \vec{l}_2 = 0$

$$\vec{k}_1 \cdot \vec{l}_1 = ac \overset{\text{set}}{=} 2\pi \Rightarrow a = \frac{2\pi}{c}$$

$$\begin{aligned} \vec{k}_2 \cdot \vec{l}_1 &= 0 \\ \vec{k}_2 \cdot l_2 &= 2\pi \end{aligned} \Longrightarrow \vec{k}_2 = \frac{2\pi}{c}\left(0, \frac{2}{\sqrt{3}}\right)$$

$$\vec{k}_1 = \frac{2\pi}{c}\left(1, \frac{1}{\sqrt{3}}\right)$$

Bifurcations on a Lattice

$$\frac{\partial u(x,t)}{\partial t} = f(u(x,t), \mu)$$

$\mu \equiv$ bif. par., $x \in \mathbb{R}^2$

$u(x,t) \in \mathbb{R}^1$

Symmetry Group: $\Gamma = E(2)$

Euclidean group of rotations, reflections and translations on a plane.

$\Longrightarrow f$ must be $E(2)$ equiv.

$$\gamma f(x, \mu) = f(\gamma u, \mu), \; \forall \gamma \in E(2)$$

$E(2)$

Translation $\equiv p \in \mathbb{R}^2$: $p \cdot u(x) = u(x-p)$

Rotation & reflection $\equiv \sigma \in O(2)$

$$\sigma \cdot u(x) = u(\sigma^{-1} x)$$

⊙ σ^{-1} is needed for the action of $E(2)$ on the space of functions to be a homomorphism.

Pf Need: $\theta(\gamma\sigma) = \theta(\gamma)\theta(\sigma)$

$\forall \gamma, \sigma \in E(2)$.

Let $\boxed{\theta(\gamma)u(x) = u(\gamma^{-1}x)}$, $\forall \gamma \in E(2)$

$$w(x) = \theta(\sigma)u(x) = u(\sigma^{-1}x)$$

$$\theta(\gamma)(\theta(\sigma)u)(x) = \theta(\gamma)w(x) = w(\gamma^{-1}x)$$

$$= u(\sigma^{-1}\gamma^{-1}x) = u((\gamma\sigma)^{-1}x)$$

$$= \theta(\gamma\sigma)u(x), \ \forall \sigma, \gamma \in E(2)$$

⊙ inverses ensure group elements act in correct order. □

ct $l \in \mathcal{L}$

$p \cdot u(x+l) = u((x+l)-p) = u((x-p)+l)$

$= u(x-p) = p \cdot u(x)$

$\Rightarrow C^\infty_{\mathcal{L}}$ is inv. under all translations

Symmetry $\Gamma(E(2))$

(PDE) $\frac{\partial}{\partial t} u(x,t) = f(u(x,t), \mu)$

} decaying/growing modes

Sol. $u(x,t) = \sum_{j=1}^{n} z_j(t) e^{i \vec{k}_j \cdot \vec{x}} + c.c.$

$z_j(t) \in \mathbb{C}$, $|\vec{k}_j| = k_c$ = critical wave number

Amp. Eq. $\frac{dz}{dt} = g(z, \mu)$

$z = (z_1, \dots, z_n) \in \mathbb{C}^n$

Stationary Trivial Sol. $z = 0 \Leftrightarrow g(0, \mu) = 0$

Def $\mathcal{H}_{\mathcal{L}}$ = Holohedry of $\mathcal{L}$

$= \{ \sigma \in O(2) ; \sigma\mathcal{L} = \mathcal{L} \}$

$\odot$ $C^\infty_{\mathcal{L}}$ is inv. under $\mathcal{H}_{\mathcal{L}}$

$\mathcal{H}_{\mathcal{L}}$ is the group of symmetries that the Amp. Eq. retains from $E(2)$ in the original PDE.