(PDE) $\frac{\partial u}{\partial t}(x,t) = f(u(x,t),\mu)$ $\qquad \vec{E}(2)$

$\Big\downarrow$ modes

$$u(x,t) = \sum_{j=1}^{n} z_j(t)\, e^{i\vec{k}_j\cdot\vec{x}} + c.c.$$

$z_j\in\mathbb{C}$, $\quad |\vec{k}_j| = k_c =$ Critical Wave Number

$\Big\downarrow$ Symmetry

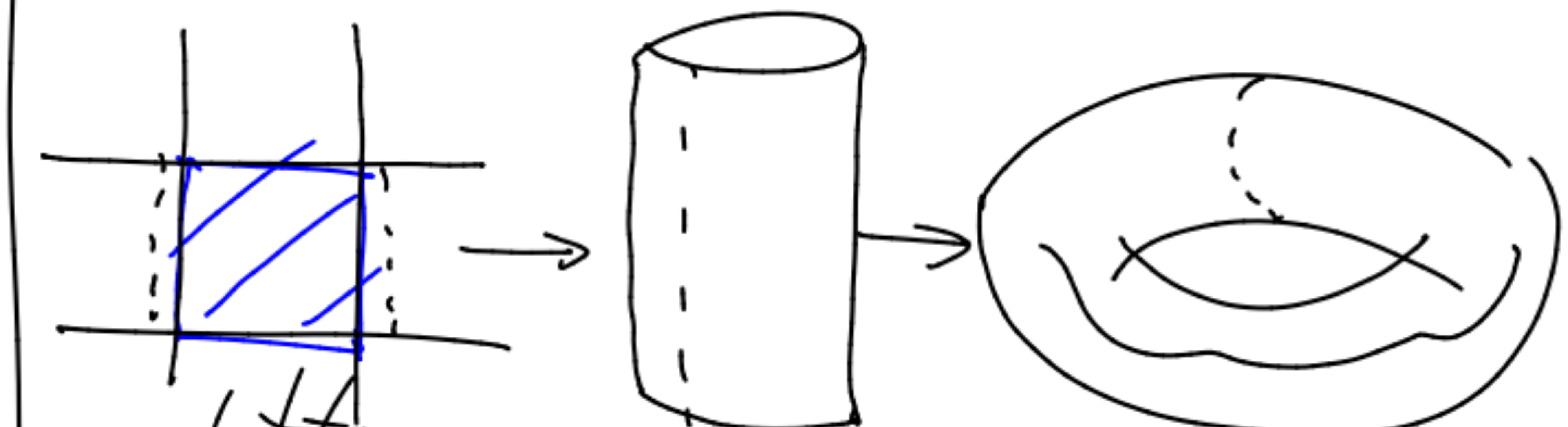
(Amp. Eq.) $\frac{dz}{dt} = g(z,\mu)$ — $\mathcal{H}_{\mathcal{L}}$

$$z=(z_1,\dots,z_n)\in\mathbb{C}^n$$

$$\mathcal{H}_{\mathcal{L}} = \{\sigma\in O(2) : \sigma\mathcal{L}=\mathcal{L}\}$$

⓪ Since we assume periodicity on $\mathcal{L}$ we can generate full patterns from the sol. in the fundamental cell.

eg.) Translations on a Square $\mathcal{L}$.



∞-lattice
none compact domain

T^2

$$T^2 \approx \underbrace{\mathbb{R}^2/\mathcal{L}}_{\text{Quotient Group}}$$

Symmetry Group

$$\Gamma = \mathcal{H}_{\square} \times T^2$$

$$\mathcal{H}_{rect} \simeq \mathcal{H}_{rhombic} \simeq \mathcal{H}_{CR} = D_2$$

$$\mathcal{H}_{square} \simeq D_4$$

$$\mathcal{H}_{hexagon} \simeq D_6$$

Steady-State Bif. on a Square $\mathcal{L}$.

$$\mathcal{L} = \mathcal{L}(\vec{l}_1, \vec{l}_2), \quad \vec{l}_1 = (1,0), \quad \vec{l}_2 = (0,1)$$

$$\mathcal{L}^* = \mathcal{L}(\vec{k}_1, \vec{k}_2), \quad \vec{k}_1 = (2\pi, 0), \quad \vec{k}_2 = (0, 2\pi)$$

$$u(x,t) = z_1(t) e^{i\vec{k}_1 \cdot \vec{x}} + z_2(t) e^{i\vec{k}_2 \cdot \vec{x}} + c.c.$$

$$= \mathrm{Re}\left\{ z_1 e^{ik_1 \cdot x} + z_2 e^{ik_2 \cdot x} \right\}, \quad \vec{x} = (x_1, x_2)$$

$$= \mathrm{Re}\left\{ z_1 e^{i2\pi x_1} + z_2 e^{i2\pi x_2} \right\}$$

Group Action

$$\begin{cases} \rho = R_{\frac{\pi}{2}} \Rightarrow M_\rho = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ or } M_\rho = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ \kappa = \text{Reflection} \Rightarrow M_\kappa = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } M_\kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{cases}$$

Textbook

$p = (p_1, p_2) \in T_2 = \text{Translation}, \quad p \cdot x \longmapsto x + p$

$$p \cdot u(x,t) = u(x+p, t) = R_e\left\{ z_1 e^{ik_1 \cdot (x+p)} + z_2 e^{ik_2 \cdot (x+p)} \right\}$$

$$= R_e\left\{ e^{2\pi p_1 i} z_1 e^{2\pi i x_1} + e^{2\pi p_2 i} z_2 e^{2\pi i x_2} \right\} \Rightarrow$$

$$\boxed{p \cdot (z_1, z_2) = (e^{2\pi i p_1} z_1, e^{2\pi i p_2} z_2)}$$

$D_4 = \{\rho, \kappa\}$

$\kappa \cdot x = (x_1, -x_2), \quad \rho \cdot x = (x_2, -x_1)$

$$\kappa \cdot u(x,t) = u(\kappa^{-1} x, t)$$

$$= R_e\left\{ z_1 e^{i2\pi x_1} + z_2 e^{-i2\pi x_2} \right\}$$

$$M_\kappa^{-1} = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} \cdot u = z_1 e^{i2\pi x_1} + \cancel{z_2} e^{-i2\pi x_2} + \bar{z}_1 e^{-i2\pi x_1} + \bar{z}_2 e^{i2\pi x_2}$$

$$\boxed{\kappa \cdot (z_1, z_2) = (z_1, \bar{z}_2)}$$

$$\cdot\, u(x,t) = u(\rho^{-1}x, t)$$

$$\rho^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow M_{\rho^{-1}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

$$\rho \cdot u = \underset{\bar{z}_2}{\underset{\updownarrow}{z_1}} e^{-i2\pi x_2} + \underset{z_1}{\underset{\updownarrow}{z_2}} e^{i2\pi x_1} +$$

$$\underset{z_2}{\underset{\updownarrow}{\bar{z}_1}} e^{i2\pi x_2} + \underset{\bar{z}_1}{\underset{\updownarrow}{\bar{z}_2}} e^{-i2\pi x_1}$$

Summary

$$\rho \cdot (z_1, z_2) = (\bar{z}_2, z_1)$$

$$k \cdot (z_1, z_2) = (z_1, \bar{z}_2)$$

$$p(z_1, z_2) = (e^{2\pi i p_1} z_1, e^{2\pi i p_2} z_2)$$

$$M_\rho = \begin{array}{c} \begin{matrix} x_1 & y_1 & x_2 & y_2 \end{matrix} \\ \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \end{array}$$

$$M_\rho g(z) = g(M_\rho z)$$

Isotropy Subgroups

$$z_1 = x_1 + y_1 i, \quad z_2 = x_2 + y_2 i$$

$$\begin{array}{ccc} (z_1, & z_2) & \xmapsto[P]{\text{choose}} (x_1, x_2) \\ \shortparallel & \shortparallel & \\ x_1 e^{2\pi i p_1}, & x_2 e^{2\pi i p_2} & \end{array}$$

$$M_k = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

$$M_p = \left[\begin{array}{cc|cc} \cos 2\pi p_1 & -\sin 2\pi p_1 & 0 & 0 \\ \sin 2\pi p_1 & \cos 2\pi p_1 & 0 & 0 \\ \hline 0 & 0 & \cos 2\pi p_2 & -\sin 2\pi p_2 \\ 0 & 0 & \sin 2\pi p_2 & \cos 2\pi p_2 \end{array}\right]$$

Rotation by $e^{2\pi i (p_1 \text{ or } p_2)}$

Sol	Σ	Pattern	dim $\underline{Fix}(\Sigma)$
$(0,0)$	$\Gamma = D_4 \times T^2$	Homogenous	0
$(a_1,0)$ or $(0,a_2)$	$\Sigma_R = SO(2) + D_2$ $(0,P_2)+D_2(\rho^2,\kappa)$	Rolls	$\underline{1}$
$(a_1,a_1)_0$	$\Sigma_S = D_4(\rho,\kappa)$	Squares	$\underline{1}$
(a_1,a_2) $a_1 \neq a_2$	$D_2(\rho^2,\kappa)$	Mixed-mode	2

$$
\begin{tikzcd}
& \Gamma = D_4 \times T^2 & \\
\begin{array}{c}\text{Squares}\\ D_4\end{array} \arrow[ur] & & \begin{array}{c}\text{Rolls}\\ SO(2)\times D_2\end{array} \arrow[ul] \\
& \begin{array}{c}\text{Mixed-Mode}\\ D_2\end{array} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathrm{Fix}(\Gamma) = 0 & \\
\mathrm{Fix}(D_4) = \mathbb{R}\{(1,1)\} \arrow[ur] & & \mathrm{Fix}(SO(2)\times D_2) = \mathbb{R}\{1,0\} \arrow[ul] \\
 & \mathrm{Fix}(D_2) = \{(a_1, a_2)\} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$