SS Bif. on a Square Lattice

$$\Gamma = D_4 \times T^2$$

$$u(x,t) = z_1(t) e^{i\vec{k}_1 \cdot \vec{x}} + z_2(t) e^{i\vec{k}_2 \cdot \vec{x}} + c.c.$$

Group action:

$$\rho \cdot (z_1, z_2) = (\bar{z}_2, z_1)$$

$$\kappa \cdot (z_1, z_2) = (z_1, \bar{z}_2)$$

$$p \cdot (z_1, z_2) = (e^{2\pi i p_1} z_1, e^{2\pi i p_2} z_2)$$

$$\begin{tikzcd}
 & \Gamma = D_4 \times T^2 & \\
\text{Squares } D_4 \arrow[ur] & & \text{Rolls } SO(2) \times D_2 \arrow[ul] \\
 & \text{Mixed-mode } D_2 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & Fix(\Gamma) = 0 & \\
Fix(D_4) = \mathbb{R}\{(1,1)\} \arrow[ur] & & Fix(SO(2) \times D_2) = \mathbb{R}\{(1,0)\} \arrow[ul] \\
 & Fix(D_2) = \{(a_1, a_2)\} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Amplitude Equations

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = g\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} g_1(z_1, z_2) \\ g_2(z_1, z_2) \end{pmatrix}$$

Translation Equiv: $p \circ g\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = g\left(p\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$

$$p \circ g\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{2\pi i p_1} g_1(z_1, z_2) \\ e^{2\pi i p_2} g_2(z_1, z_2) \end{pmatrix}$$

want

$$g\left(p\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = \begin{pmatrix} g_1(e^{2\pi i p_1} z_1, e^{2\pi i p_2} z_2) \\ g_2(e^{2\pi i p_1} z_1, e^{2\pi i p_2} z_2) \end{pmatrix}$$

Set $p_1 = 0$

$$g_1(z_1, e^{2\pi i p_2} z_2) = g_1(z_1, z_2)$$

$$\Downarrow$$

$$g_1(z_1, z_2) = h_1(z_1, |z_2|^2)$$

Set $p_2 = 0$

$$g_1(e^{2\pi i p_1} z_1, z_2) = e^{2\pi i p_1} g_1(z_1, z_2)$$

$$\Downarrow$$

$$g_1(z_1, z_2) = P_1(|z_1|^2, |z_2|^2) z_1$$

$$LHS = g_1(e^{2\pi i p_1} z_1, z_2) = P_1(|z_1|^2, |z_2|^2) \cancel{e^{2\pi i p_1}} z_1$$

$$RHS = e^{2\pi i p_1} g_1(z_1, z_2) = \cancel{e^{2\pi i p_1}} P_1(|z_1|^2, |z_2|^2) z_1$$

Similary

$g_2(z_1, z_2) = P_2(|z_1|^2, |z_2|^2) z_2$

Reflection Equiv: $k \cdot g\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = g\left(k \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$

$k \cdot g\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} g_1(z_1, z_2) \\ \overline{g_2(z_1, z_2)} \end{pmatrix}$

want

$g\left(k\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = \begin{pmatrix} g_1(z_1, \bar{z}_2) \\ g_2(z_1, \bar{z}_2) \end{pmatrix}$

$g_1(z_1, z_2) = g_1(z_1, \bar{z}_2) \Rightarrow$ no extra info

$g_2(z_1, \bar{z}_2) = \overline{g_2(z_1, z_2)}$

$P_2(|z_1|^2, |z_2|^2)\bar{z}_2 \overset{\text{want}}{=} \overline{P_2(|z_1|^2, |z_2|^2)}\,\bar{z}_2$

$P_2 = \overline{P_2} \Rightarrow P_2 \in \mathbb{R}$

Similarly, $P_1 \in \mathbb{R}$

So far:

$$g = \begin{pmatrix} P_1(|z_1|^2, |z_2|^2) z_1 \\ P_2(|z_1|^2, |z_2|^2) z_2 \end{pmatrix}$$

$P_1, P_2 \in \mathbb{R}$

Rotation Equiv. $\rho\cdot g\begin{pmatrix}z_1\\ z_2\end{pmatrix} = g\left(\rho\cdot\begin{pmatrix}z_1\\ z_2\end{pmatrix}\right)$

$$\rho\cdot g\begin{pmatrix}z_1\\ z_2\end{pmatrix} = \begin{pmatrix}p_2(|z_1|^2,|z_2|^2)\,\bar{z}_2\\ p_1(|z_1|^2,|z_2|^2)\,z_1\end{pmatrix}$$

want

$$g\circ\rho\begin{pmatrix}z_1\\ z_2\end{pmatrix} = \begin{pmatrix}p_1(|z_2|^2,|z_1|^2)\,\bar{z}_2\\ p_2(|z_2|^2,|z_1|^2)\,z_1\end{pmatrix}$$

$$\Longrightarrow\ p_1 = p_2$$

(i) Trivial Sol: $z_1 = z_2 = 0$

(ii) x_1-rolls : $|z_1|^2 = \frac{\mu}{a_1}$, $z_2 = 0$

(iii) x_2-rolls : $|z_2|^2 = \frac{\mu}{a_1}$, $z_1 = 0$

(iv) Squares : $|z_1|^2 = |z_2|^2 = \frac{\mu}{a_1+a_2}$

<u>Summary</u>

$$g\begin{pmatrix}z_1\\ z_2\end{pmatrix} = \begin{pmatrix}p(|z_1|^2,|z_2|^2)\,z_1\\ p(|z_1|^2,|z_2|^2)\,z_2\end{pmatrix},\ p\in\mathbb{R}$$

$$\frac{d}{dt}\begin{pmatrix}z_1\\ z_2\end{pmatrix} = g\begin{pmatrix}z_1\\ z_2\end{pmatrix}$$

Taylor

$$\boxed{\begin{aligned}\dot{z}_1 &= \mu z_1 - a_1|z_1|^2 z_1 - a_2|z_2|^2 z_1\\ \dot{z}_2 &= \mu z_2 - a_1|z_2|^2 z_2 - a_2|z_1|^2 z_2\end{aligned}}$$

Rolls: $g(z_1, z_2, \lambda)\Big|_{\substack{z_1 = a_1 \\ z_2 = 0}}$

$$g(a_1, 0, \lambda) = \begin{pmatrix} P(a_1^2, 0, \lambda)\, a_1 \\ 0 \end{pmatrix} \overset{set}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[P_{a_1}(0)\, a_1^2 + P_\lambda(0)\, \lambda + \ldots \right] a_1 \overset{set}{=} 0$$

$$\boxed{\lambda = -\frac{P_{a_1}(0)}{P_\lambda(0)}\, a_1^2 + \ldots}$$

Pitchfork Bif. to Rolls

Squares $g(z_1, z_2, \lambda)\Big|_{\substack{z_1 = a_1 \\ z_2 = a_1}}$

$$g(a_1, a_1, \lambda) = \begin{pmatrix} P(a_1^2, a_1^2, \lambda)\, a_1 \\ P(a_1^2, a_1^2, \lambda)\, a_1 \end{pmatrix} \overset{set}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[P_{a_1}(0) + P_{a_2}(0) \right] a_1^2 + P_\lambda(0) + \ldots = 0$$

$$\boxed{\lambda = -\frac{P_{a_1}(0) + P_{a_2}(0)}{P_\lambda(0)}\, a_1^2}$$

Pitchfork Bif. to Squares

Stability of Rolls and Squares

$$(dg)_{(z_1,z_2)}(w_1,w_2) = g_{z_1}w_1 + g_{\bar z_1}\bar w_1 + g_{z_2}w_2 + g_{\bar z_2}\bar w_2$$

$$g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} P(|z_1|^2,|z_2|^2)\, z_1 \\ P(|z_1|^2,|z_2|^2)\, z_2 \end{pmatrix}$$

Squares: $(z_1,z_2) = (a,a) \in \mathbb{R}$

Chain Rule: $w = f(u,v)$, $u = g(x,y)$, $v = h(x,y)$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

$$g_{1,z_1} = P + \frac{\partial P}{\partial z_1} z_1 = P + \left[\frac{\partial P}{\partial u}\frac{\partial u}{\partial z_1} + \frac{\partial P}{\partial v}\underbrace{\frac{\partial v}{\partial z_1}}_{0}\right] z_1$$

$u = z_1 \bar z_1$, $v = z_2 \bar z_2$

$$= P + \left[\frac{\partial P}{\partial u}\bar z_1 + 0\right] z_1 = P + P_1 |z_1|^2$$

$$g_{2,z_1} = \left[\frac{\partial P}{\partial u}\underbrace{\frac{\partial u}{\partial z_2}}_{0} + \frac{\partial P}{\partial v}\frac{\partial v}{\partial z_2}\right] z_1 = \left[\frac{\partial P}{\partial v}\bar z_2\right] z_1$$

$$= P_2 \bar z_2 z_1$$

$$g_{1,z_1} = P + P_1|z_1|^2 = \underbrace{P(a^2,a^2)}_{\to 0 \text{ at bif}} + P_1(a^2,a^2)a^2$$

$$g_{1,z_2} = P_2 \bar{z}_2 z_1 = P_2(a^2,a^2)a^2$$

$$g_{1,\bar{z}_1} = P_1 z_1^2 = P_1(a^2,a^2)a^2$$

$$g_{1,\bar{z}_2} = P_2 z_2 z_1 \Big|_{(a,a)} = P_2(a^2,a^2)a^2$$

$$g_{2,z_1} = P_2 \bar{z}_1 z_2 = P_2(a^2,a^2)a^2$$

$$g_{2,z_2} = P + P_1|z_2|^2 = P_1(a^2,a^2)a^2$$

$$g_{2,\bar{z}_1} = P_2 z_1 z_2 = P_2(a^2,a^2)a^2$$

$$g_{2,\bar{z}_2} = P_1 z_2^2 \Big|_{(a,a)} = P_1(a^2,a^2)a^2$$

$$(dg)_{(z_1,z_2)}(w_1,w_2) = a^2\left[w_1\begin{pmatrix}P_1\\P_2\end{pmatrix} + \bar{w}_1\begin{pmatrix}P_1\\P_2\end{pmatrix} + w_2\begin{pmatrix}P_2\\P_1\end{pmatrix} + \bar{w}_2\begin{pmatrix}P_2\\P_1\end{pmatrix}\right]$$

Let $w_1 = w_2 = 1$ since $a(1,1)$ is inv.

$$(dg)_{(z_1,z_2)}(w_1=1,w_2=1) = a^2\begin{bmatrix}2P_1+2P_2\\2P_2+2P_1\end{bmatrix} = 2(P_1+P_2)a^2\begin{pmatrix}1\\1\end{pmatrix}$$

e-val. λ

Let $w_1 = 1,\ w_2 = -1$

$$dg\big|_{(z_1,z_2)}(1,-1) = a^2\begin{bmatrix} 2P_1 - 2P_2 \\ 2P_2 - 2P_1 \end{bmatrix}$$

$$= \underbrace{2(P_1 - P_2)\,a^2}_{\text{e-val } \lambda_2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Let $w_1 = iV_1,\ w_2 = iV_2.$

$$dg\big|_{(z_1,z_2)}(iV_1, iV_2) = a^2\left[\cancel{iV_1\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}} - \cancel{iV_1\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}} + \cancel{iV_2\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}} - \cancel{iV_2\begin{pmatrix} P_2 \\ P_1 \end{pmatrix}}\right] = \underbrace{0}_{\lambda_3}$$

⊙ There is one more zero e-val forced by symmetry.

<u>Summary</u>.

Eigenvalues of (dg) at squares are:

$$\frac{0\,(2),\quad 2(P_1+P_2)a^2,\quad 2(P_1-P_2)a^2}{\mathbb{C}^4 = \underbrace{\overbrace{Fix(D_4)}^{\mathbb{R}\{(1,1)\}}}_{\text{natural action}} \oplus \underbrace{\overbrace{V_1}^{\mathbb{R}\{(1,-1)\}}}_{\text{nontrivial}} \oplus \underbrace{\overbrace{V_2}^{\mathbb{R}\{(iV_1,iV_2)\}}}_{\text{zero e-val}}}$$