SS Bif. on a Hexagonal Lattice

$\Gamma = D_6 \times T^2$

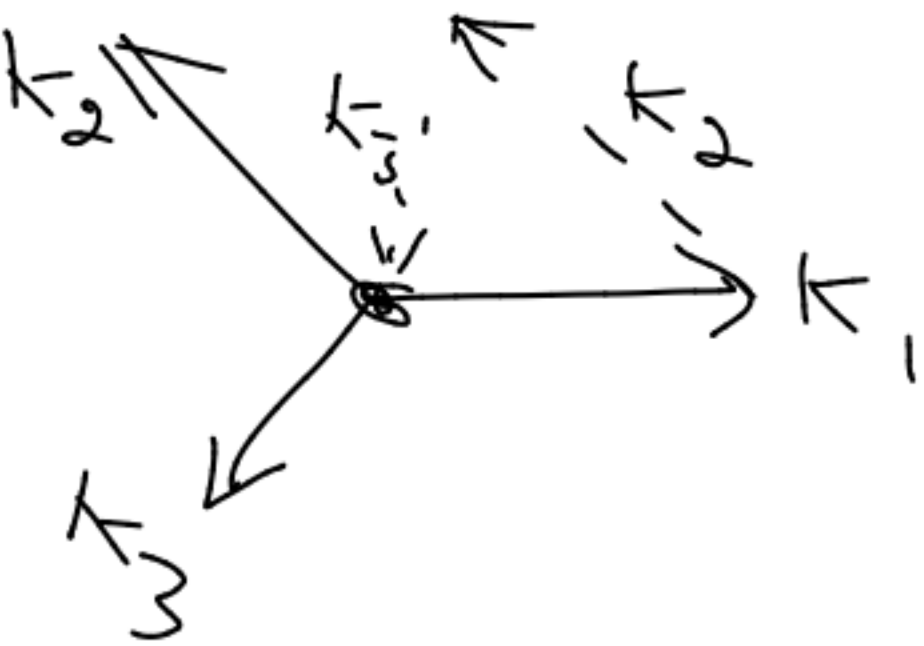
$\mathcal{L} \equiv \mathcal{L}\{\vec{\ell}_1, \vec{\ell}_2\}$

$\vec{\ell}_1 = \left(1, \frac{1}{\sqrt{3}}\right), \ \vec{\ell}_2 = \left(0, \frac{2}{\sqrt{3}}\right)$

$\mathcal{L}^* = \mathcal{L}(\vec{k}_1, \vec{k}_2)$

$\vec{k}_1 = (1,0), \ \vec{k}_2 = \frac{1}{2}(-1, \sqrt{3}), \ \vec{k}_3 = \frac{1}{2}(-1, -\sqrt{3})$

$k_1 + k_2 + k_3 = 0$ (resonance cond.)



$$u(x,t) = \sum_{j=1}^{3} z_j(t) e^{i\vec{k}_j \cdot \vec{x}} + c.c.$$

Group Action

$\rho \equiv R_{\frac{2\pi}{3}}, \ p \in T^2 \equiv$ Translations

$m_0 \equiv$ Reflection through origin (rotation by π)

$m_v \equiv$ Reflection on a vertical plane

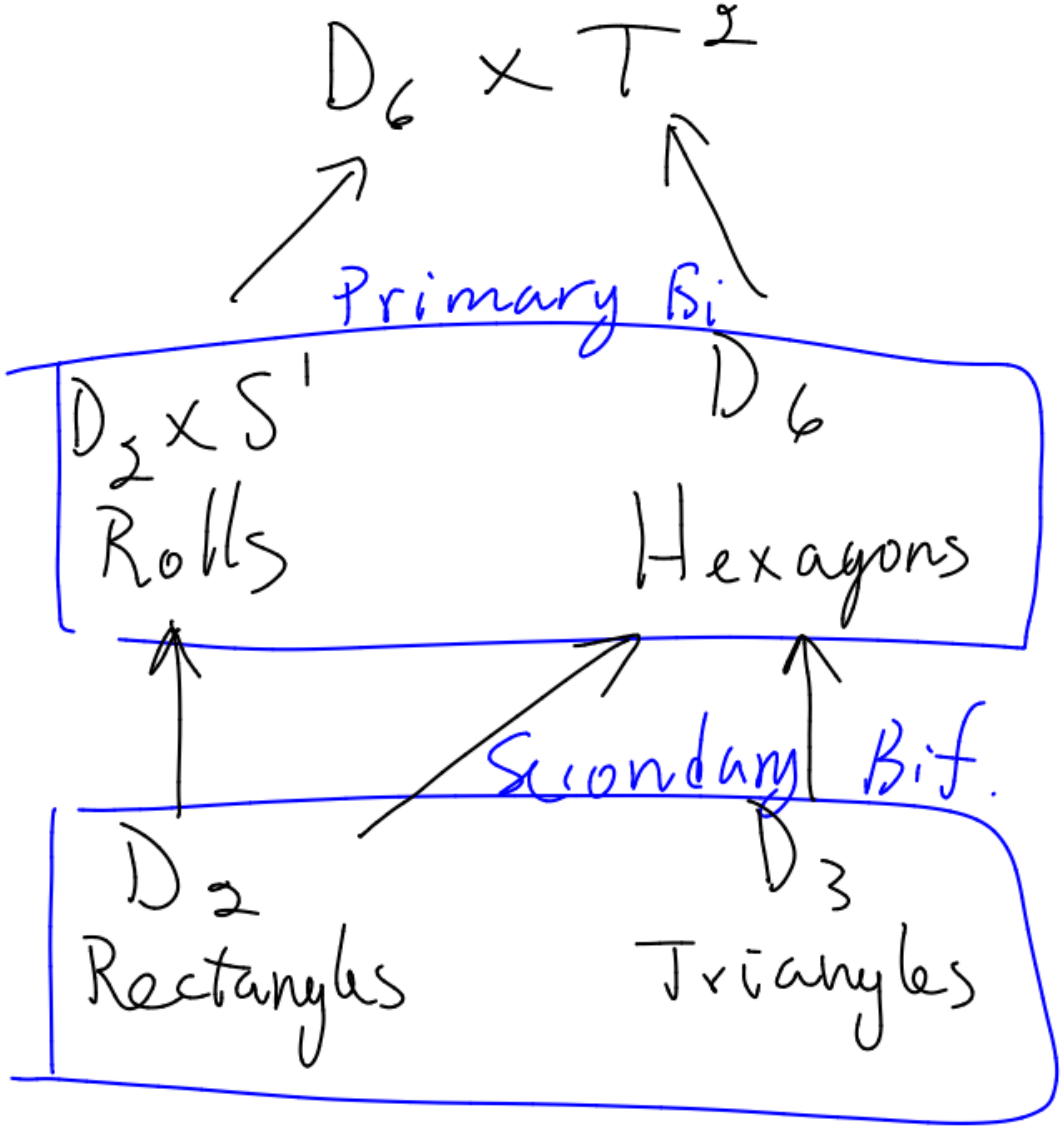
$\rho \cdot (z_1, z_2, z_3) = (z_3, z_1, z_2)$

$m_0 \cdot (z_1, z_2, z_3) = (\bar{z}_1, \bar{z}_2, \bar{z}_3)$

$m_v (z_1, z_2, z_3) = (z_1, z_3, z_2)$

$p \cdot (z_1, z_2, z_3) = (e^{-ik_1 \cdot p} z_1, e^{-ik_2 \cdot p} z_2, e^{-ik_3 \cdot p} z_3)$

ol	Orbit	Isotropy	$\dim \mathrm{Fix}(\Sigma)$
Trivial	$(0,0,0)$	$D_6 \times T^2$	0
Rolls	$(x,0,0)$	$D_2 \times S^1$	1
Hexagons Up $x>0$ Down $x<0$	(x, x, x)	D_6	1
Rectangles Up $x_1>0$ Down $x_1<0$	(x_1, x_2, x_2) Patchwork Quilt $x_1 = 0$	D_2	2
Triangles Regular	$(x_1+y_1 i, x_1+y_1 i, x_1+y_1 i)$ $x_1 = 0$	D_3	2


$D_6 \times T^2$
Primary Bi
$D_2 \times S^1$
Rolls
D_6
Hexagons
Secondary Bif.
D_2
Rectangles
D_3
Triangles

Normal Forms (Cubic Order)

$$\dot{z}_1 = \mu z_1 + a\bar{z}_2\bar{z}_3 - b|z_1|^2 z_1 - c(|z_2|^2 + |z_3|^2) z_1$$
$$\dot{z}_2 = \mu z_2 + a\bar{z}_3\bar{z}_1 - b|z_2|^2 z_2 - c(|z_3|^2 + |z_1|^2) z_2$$
$$\dot{z}_3 = \mu z_3 + a\bar{z}_1\bar{z}_2 - b|z_3|^2 z_3 - c(|z_1|^2 + |z_2|^2) z_3$$

Solutions

(i) Trivial: $z_1 = z_2 = z_3 = 0$

(ii) Rolls: $z_1 = \sqrt{\frac{\mu}{b}}, \; z_2 = z_3 = 0$

(iii) Hexagons: $\mathrm{Re}(z_1) = \mathrm{Re}(z_2) = \mathrm{Re}(z_3) = R_0$

where $\mu + aR_0 - (b+2c)R_0^2 = 0$

$\mathrm{Im}(z_j) = 0$

v) Rectangles

$$Re(z_1) = -\frac{a}{b-c}$$

$$Re(z_2) = Re(z_3) = \pm\sqrt{\left(\frac{1}{b+c}\right)\left(\mu - \frac{a^2 b}{(b-c)^2}\right)}$$

$$Im(z_i) = 0$$

⑤ Not all sol. are equilibria of NF (truncated at third order). Some may require h.o.t.

Stability

Trivial Sol. $z_1 = z_2 = z_3 = 0$ $\begin{cases} \text{stable}, & \mu < 0 \\ \text{unstable}, & \mu > 0 \end{cases}$

Rolls

$z_1 = R_0(1+\delta z_1)$, $R_0 = \sqrt{\frac{\mu}{b}}$

$z_2 = 0 + \delta z_2$, $|\delta z_j| \ll 1$

$z_3 = 0 + \delta z_3$,

Linearizing:

$$\dot{z}_1 = R_0 \frac{d}{dt}(\delta z_1) = \mu R_0 (1+\delta z_1) - \underbrace{bR_0(1+\delta z_1)}_{z_1} \underbrace{R_0(1+\delta \bar{z}_1)}_{\bar{z}_1} \underbrace{R_0(1+\delta z_1)}_{z_1}$$

$$\dot{(\delta z_1)} = \Big[\mu - \underbrace{\cancel{b}R_0^2}_{\frac{\mu}{\cancel{b}}}(1+\delta z_1)(1+\delta\bar{z}_1)\Big]\cancel{R_0}(1+\delta z_1)$$

$$= \mu\Big[1-(1+\delta\bar{z}_1+\delta z_1+\underbrace{\delta z_1\delta\bar{z}_1}_{h.o.t.})\Big](1+\delta z_1)$$

$$\frac{d}{dt}(\delta z_1) = -\mu(\delta z_1+\delta\bar{z}_1)$$

$$\Downarrow$$

$$\frac{d}{dt}(\delta\bar{z}_1) = -\mu(\delta z_1+\delta\bar{z}_1)$$

$$\frac{d}{dt}(\delta z_1+\delta\bar{z}_1) = -2\mu(\delta z_1+\delta\bar{z}_1) \qquad \sigma_1$$

$$\frac{d}{dt}(\delta z_1-\delta\bar{z}_1) = 0 \qquad \sigma_2$$

$$\frac{d}{dt}(\delta z_2) = \ldots = \mu\left(1-\frac{c}{b}\right)\delta z_2 + a\sqrt{\frac{\mu}{b}}\,\delta\bar{z}_3$$

$$\frac{d}{dt}(\delta z_3) = \mu\left(1-\frac{c}{b}\right)\delta z_3 + a\sqrt{\frac{\mu}{b}}\,\delta \bar{z}_2$$

$$\Downarrow$$

$$\sigma_{3-6} = \mu\left(1-\frac{c}{b}\right) \pm a\sqrt{\frac{\mu}{b}} \quad \left(\begin{matrix}\text{twice}\\ \text{each}\end{matrix}\right)$$

Summary: (neutrally)

Rolls are linearly stable if $\mu > 0$ $(\sigma_1 < 0)$ and $b-c<0$ and $\sqrt{\frac{\mu}{b}} > -\frac{a}{b-c}$ $(\sigma_{3-6} < 0)$

Hexagons

$$z_1 = R_0(1+\delta z_1)$$

$$z_2 = R_0(1+\delta z_2)$$

$$z_3 = R_0(1+\delta z_3)$$

$$\Downarrow$$

$$\sigma_1 = -3aR_0, \quad \sigma_{2,3} = 0$$

$$\sigma_4 = aR_0 - 2(b+2c)R_0^2$$

$$\sigma_{5,6} = -2aR_0 - 2(b-c)R_0^2 \quad (\text{twice})$$

$\xRightarrow{\text{stable}}$ $R_0 > 0$, $aR_0 - 2(b+2c)R_0^2 < 0$

$-2aR_0 - 2(b-c)R_0^2 < 0$

Bif. Diagram