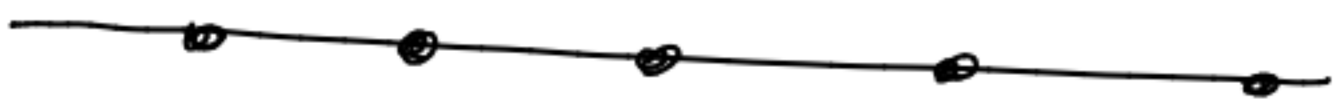
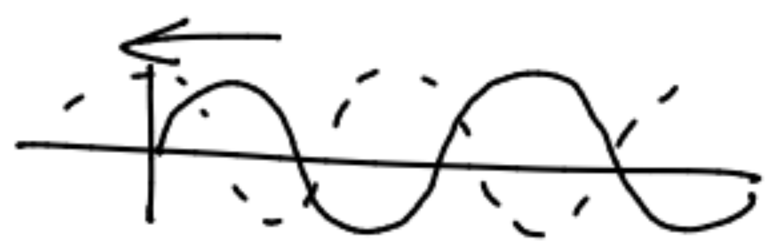
Hopf Bif. on a 1D Lattice

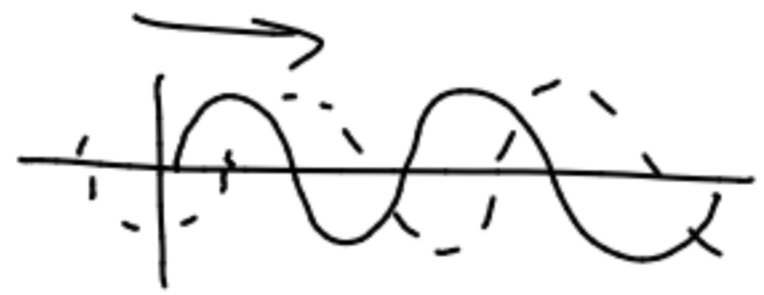


$$u(x,t) = \underbrace{z_1(t)e^{i(x-t)}}_{\text{wave travelling to the right}} + \underbrace{z_2(t)e^{-i(x+t)}}_{\text{wave traveling to the left}} + c.c. + h.o.t.$$

$z_1 = 0,\ z_2 \neq 0$: $\overleftarrow{TW}$



$z_1 \neq 0,\ z_2 = 0$: $\overrightarrow{TW}$



$z_1 = \pm z_2$: SW

Assume:

Reflection: $\kappa : x \mapsto -x$

$$\kappa \cdot (z_1, z_2) = (z_2, z_1)$$

Translation in x

$p : x \mapsto x + p$

$$p \cdot (z_1, z_2) = \left(e^{-ip} z_1,\ e^{ip} z_2\right)$$

6 Effectively, the system has $O(2)$ symmetry $\Rightarrow$ Hopf Bif. with $\Gamma = O(2)$

Recall: Hopf bif. with $\Gamma = O(2)$ | Lattice

Recall: Hopf bif. with $\Gamma = O(2)$	Lattice
$\varphi \cdot (z_1, z_2) = (e^{-i\varphi} z_1, e^{i\varphi} z_2)$	$p \cdot (z_1, z_2) = (e^{-ip} z_1, e^{ip} z_2)$
$\kappa \cdot (z_1, z_2) = (z_2, z_1)$	$A \cdot (z_1, z_2) = (z_2, z_1)$
$\theta \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$	Phase shift ?

θ System is unchanged under a shift of the origin in time (experiment does not depend on whether it is time t_1 or t_2)

$$\Rightarrow t \longmapsto t + \theta \Rightarrow (z_1, z_2) \longmapsto (e^{i\theta} z_1, e^{i\theta} z_2)$$

Right-TW: $z_1 e^{i(x-t)} + c.c.$

$$\Sigma = \widetilde{SO}(2)$$

$$\{\theta,\theta\} : (x,t) \mapsto (x+\theta, t+\theta)$$

Left-TW: $z_2 e^{-i(x+t)}$

$$\Sigma = \widetilde{SO}(2)$$

$$\{\theta,\theta\} : (x,t) \mapsto (x-\theta, t+\theta)$$

SW: $z_1 e^{i(x-t)} + z_2 e^{-i(x+t)} + c.c.$

$$\Sigma = \mathbb{Z}_2 \times \mathbb{Z}_2^c$$

Amplitude Eq:

$$\dot{z}_1 = (\mu + \nu i) z_1 - (\alpha + \beta i)|z_1|^2 z_1 - (\gamma + \delta i)|z_2|^2 z_1,$$

$$\dot{z}_2 = \underbrace{(\mu + \nu i)}_{\text{detuning par.}} z_2 - (\alpha + \beta i)|z_2|^2 z_2 - (\gamma + \delta i)|z_1|^2 z_2$$

At bif point $\mu = 0$, we want a 2π-per. sol. $\Longrightarrow \nu(\mu = 0) = 0$

ⓑ Away from $\mu = 0$, $\nu \neq 0$

Rescale: $z_j \mapsto z_j e^{i\nu t}$, $j=1,2$

$$\dot{z}_1 = \mu z_1 - (\alpha+\beta i)|z_1|^2 z_1 - (\gamma+\delta i)|z_2|^2 z_1$$

$$\dot{z}_2 = \mu z_2 - (\alpha+\beta i)|z_2|^2 z_2 - (\gamma+\delta i)|z_1|^2 z_2$$

<u>Polar Coord.</u>

Let $z_1 = R e^{i\phi}$ $\qquad R, S > 0$

$z_2 = S e^{i\psi}$, $\qquad 0 \le \phi, \psi \le 2\pi$

$$\dot{R} = \mu R - \alpha R^3 - \gamma S^2 R \stackrel{set}{=} 0$$

$$\dot{\phi} = -\beta R^2 - \delta S^2 = c$$

$$\dot{S} = \mu S - \alpha S^3 - \gamma R^2 S = 0$$

$$\dot{\psi} = -\beta S^2 - \delta R^2 = 0$$

(i) Trivial Sol $z_1 = z_2 = 0$ OR $R = S = 0$

(ii) Right-TW: $S = 0$ $(z_2 = 0)$, $R = \sqrt{\frac{\mu}{\alpha}}$, $\dot{\phi} = -\beta \frac{\mu}{\alpha}$

(iii) Left-TW: $R = 0$ $(z_1 = 0)$, $S = \sqrt{\frac{\mu}{\alpha}}$, $\dot{\psi} = -\beta \frac{\mu}{\alpha}$

(iv) SW: $R = S = \sqrt{\frac{\mu}{\alpha + \gamma}}$, $\dot{\phi} = \dot{\psi} = -\frac{\mu(\beta + \delta)}{\alpha + \gamma}$

Stability

Remarks.

1. Amp. eq. decouple
2. $\dot{\phi}$ and $\dot{\psi}$ depend only on R, S
3. If R and S are stable to perturbations then so will ϕ & ψ

Right-TW

Let $R = \sqrt{\frac{\mu}{\alpha}}\,(1+r)$, $|r|, |s| \ll 1$

$S = 0 + s$ small pert.

Linearization

$$\dot{R} = \mu R - \alpha R^3 - \gamma S^2 R$$
$$\dot{S} = \mu S - \alpha S^3 - \gamma R^2 S$$

$$J = \begin{bmatrix} \mu - 3\alpha R^2 - \gamma S^2 & -2\gamma S R \\ -2\gamma R S & \mu - 3\alpha S^2 - \gamma R^2 \end{bmatrix} \Bigg|_{\left(R^2 = \frac{\mu}{\alpha},\, S = 0\right)}$$

$$J = \begin{bmatrix} -2\mu & 0 \\ 0 & \mu\left(1 - \frac{\gamma}{\alpha}\right) \end{bmatrix}$$

e-vals: $\sigma_1 = -2\mu$, $\sigma_2 = \mu\left(1 - \frac{\gamma}{\alpha}\right)$

Stability: $\mu > 0 \Rightarrow \alpha > 0 \Rightarrow \gamma > \alpha$

Left-TW : Same as Right-TW.

$$\Downarrow$$

Exchange $R \Longleftrightarrow S$

SW : $(R^*, S^*) = \left(\sqrt{\frac{\mu}{\alpha+\gamma}}, \sqrt{\frac{\mu}{\alpha+\gamma}}\right)$

$$J\Big|_{(R^*,S^*)} = -\frac{1}{(\alpha+\gamma)}\begin{bmatrix} 2\alpha\mu & 2\gamma\mu \\ 2\alpha\mu & 2\gamma\mu \end{bmatrix} \Longrightarrow \text{Stability when}$$

$$\Rightarrow \begin{bmatrix} \dot{r} \\ \dot{s} \end{bmatrix} = -\frac{2}{(\alpha+\gamma)}\begin{bmatrix} \alpha\mu & \gamma\mu \\ \alpha\mu & \gamma\mu \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}$$

$$\frac{d}{dt}(r+s) = -2\mu(r+s)$$

$$\frac{d}{dt}(r-s) = -2\mu\frac{(\alpha-\gamma)}{(\alpha+\gamma)}(r-s)$$

$\mu > 0$ and $\alpha > \gamma$

Similar to D_4 Symm. Breaking

$$\left.\begin{array}{lcl} TW & \sim & \text{Rolls} \\ SW & \sim & \text{Squares} \\ \alpha & \sim & a_1 \\ \gamma & \sim & a_2 \end{array}\right\} \text{Fig. 4.10}$$