Full System

Let $\tilde{u}(x,t) = \hat{u}\, \boxed{e^{i\vec{k}\cdot\vec{x}+\sigma t}} + c.c.$

$\tilde{v}(x,t) = \underbrace{\hat{v}}_{\text{complex constants}}\, \boxed{e^{i\underset{\text{wave vector}}{\vec{k}}\cdot\vec{x}+\sigma t}} + c.c.$

Fourier Modes

Substituting into the Linearized System:

$$\sigma\hat{u}\cancel{e^{i\vec{k}\cdot\vec{x}+\sigma t}} = a\hat{u}\cancel{e^{i\vec{k}\cdot\vec{x}+\sigma t}} - b\hat{v}\cancel{e^{i\vec{k}\cdot\vec{x}+\sigma t}} + \overbrace{D_u\nabla^2\left(\hat{u}e^{i\vec{k}\cdot\vec{x}+\sigma t}\right)}^{-|\vec{k}|^2 D_u \hat{u}\cancel{e^{i\vec{k}\cdot\vec{x}+\sigma t}}}$$

$$\sigma\hat{v}\cancel{e^{i\vec{k}\cdot\vec{x}+\sigma t}} = c\hat{u}\cancel{e^{i\vec{k}\cdot\vec{x}+\sigma t}} - d\hat{v}\cancel{e^{i\vec{k}\cdot\vec{x}+\sigma t}} + \underbrace{D_v\nabla^2\left(\hat{v}e^{i\vec{k}\cdot\vec{x}+\sigma t}\right)}_{-|\vec{k}|^2 D_v \hat{v}\cancel{e^{i\vec{k}\cdot\vec{x}+\sigma t}}}$$

$$\sigma\hat{u} = a\hat{u} - b\hat{v} - |\vec{k}|^2 D_u \hat{u}$$

$$\sigma\hat{v} = c\hat{u} - d\hat{v} - |\vec{k}|^2 D_v \hat{v}$$

$$\sigma\begin{bmatrix}\hat{u}\\ \hat{v}\end{bmatrix}=\underbrace{\begin{bmatrix}a & -b\\ c & -d\end{bmatrix}}_{J}\underbrace{\begin{bmatrix}\hat{u}\\ \hat{v}\end{bmatrix}}_{\vec{w}}-|k|^2\underbrace{\begin{bmatrix}D_u & 0\\ 0 & D_v\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}\hat{u}\\ \hat{v}\end{bmatrix}}_{\vec{w}}$$

Linear System

unknowns: $(\hat{u},\hat{v})$

$$(\underbrace{\sigma I - J + |\vec{k}|^2 D}_{A})\vec{W}=\vec{0}$$

$$X = 0$$

Trivial Sol. $\vec{W}=\vec{0}$ (A is inv.)

∞-many sol. when A is singular

$$\Rightarrow \det(\sigma I - J + |\vec{k}|^2 D) = 0, \quad \text{let } k^2 = |\vec{k}|^2$$

$$A=\begin{bmatrix}\sigma & 0\\ 0 & \sigma\end{bmatrix}-\begin{bmatrix}a & -b\\ c & -d\end{bmatrix}+\begin{bmatrix}k^2D_u & 0\\ 0 & k^2D_v\end{bmatrix}=\begin{bmatrix}\sigma-a+k^2D_u & b\\ -c & \sigma+d+k^2D_v\end{bmatrix}$$

$$\boxed{\sigma^2+\sigma(D_uk^2+D_vk^2-a+d)+(D_uk^2-a)(D_vk^2+d)+bc=0}$$

② Steady state (U_0, V_0) $\begin{cases} \text{stable if } Re(\sigma) < 0 \\ \text{unstable if } Re(\sigma) > 0 \end{cases}$

<u>Case I</u>: No Diffusion: $D_u = D_v = 0$

Recall: $a < d \Rightarrow -a + d \lessgtr 0$

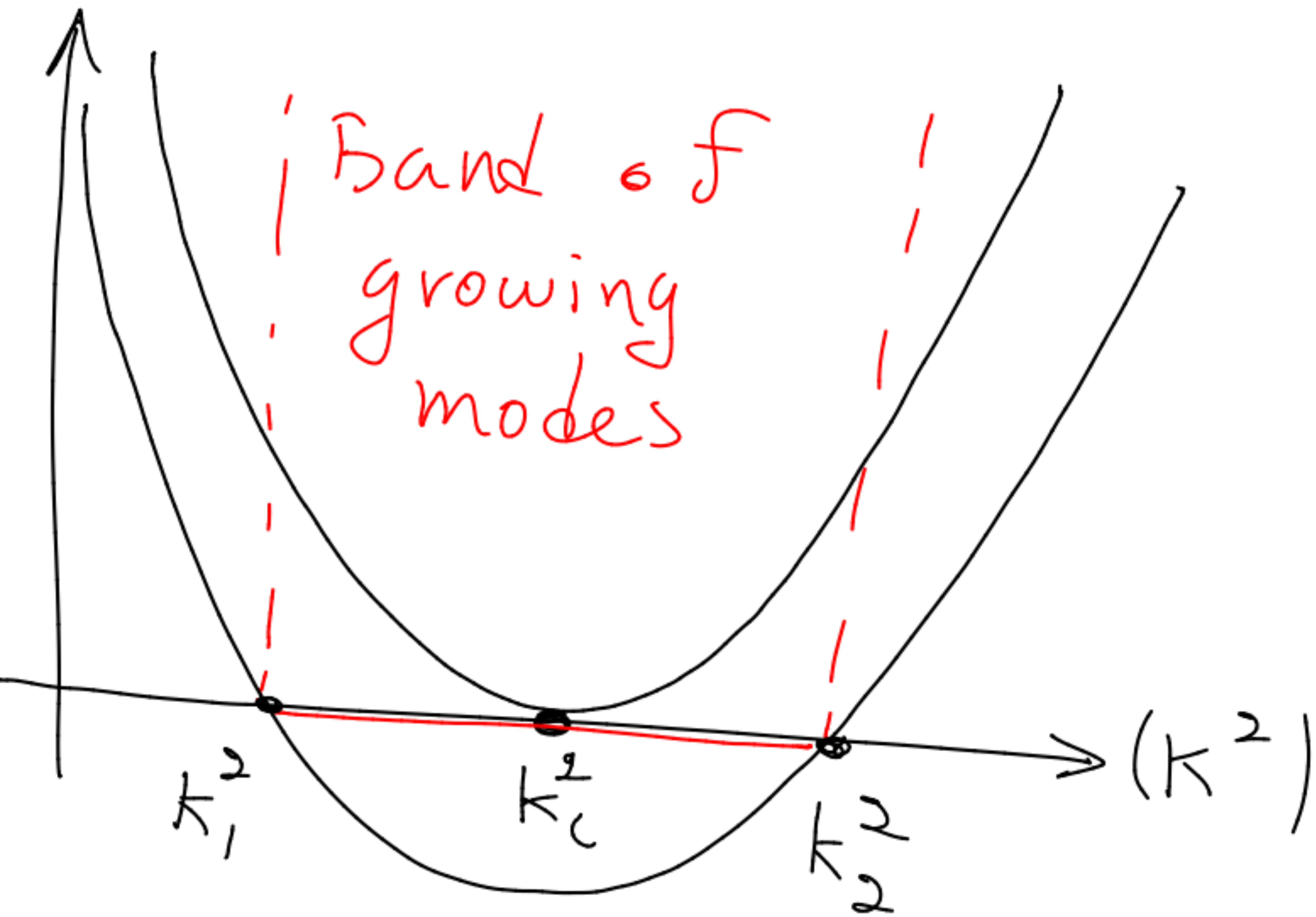
Since $(D_u + D_v)k^2 > 0$, the sum of the roots is:
$$-(D_u k^2 + D_v k^2 - a + d) < 0$$

$Re(\sigma) > 0$ can only happen if either

$(D_u k^2 + D_v k^2 - a + d) < 0$ (not possible)

OR $\underbrace{h(k^2)}_{\text{product of roots}} = (D_u k^2 - a)(D_v k^2 + d) + bc < 0 \Big\}$ quadratic in (k^2)

$$h(k^2) = D_u D_v (k^2)^2 + (dD_u - aD_v)(k^2) - ad + bc$$



$h(k^2) < 0 \Rightarrow$ inhibitor must diffuse faster for a spatial pattern to be created from the perturbation

$$h' = 2D_u D_v k^2 + (dD_u - aD_v) \overset{set}{=} 0$$

$$\boxed{k_c^2 = \frac{1}{2}\left(\frac{a}{D_u} - \frac{d}{D_v}\right)}$$

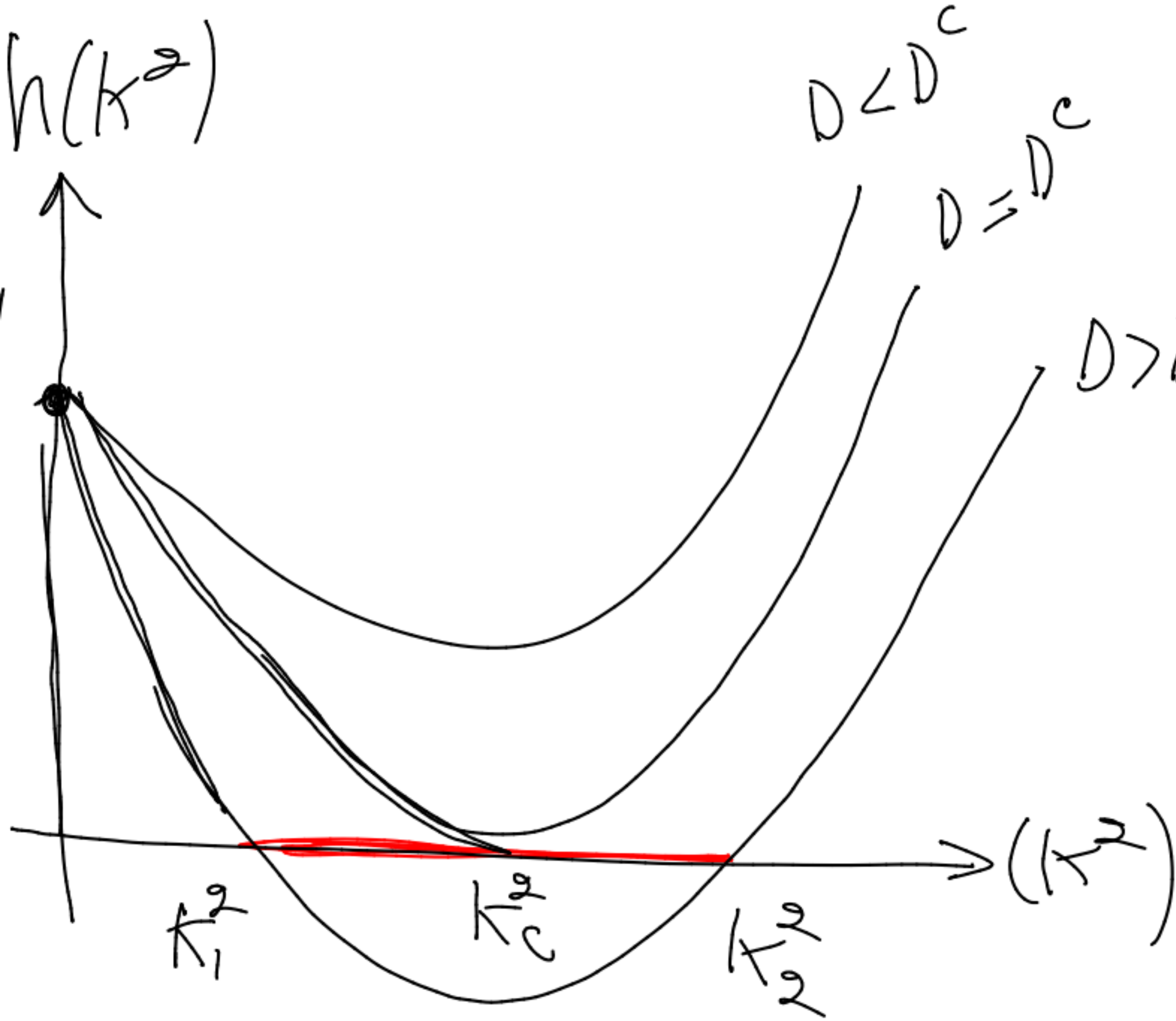
$$\min = h(k_c^2) = \ldots = \boxed{-\frac{1}{4D_uD_v}(dD_u - aD_v)^2 \underbrace{-ad+bc}_{|J|}} = \boxed{|J| - \frac{1}{4D_uD_v}(dD_u - aD_v)^2}$$

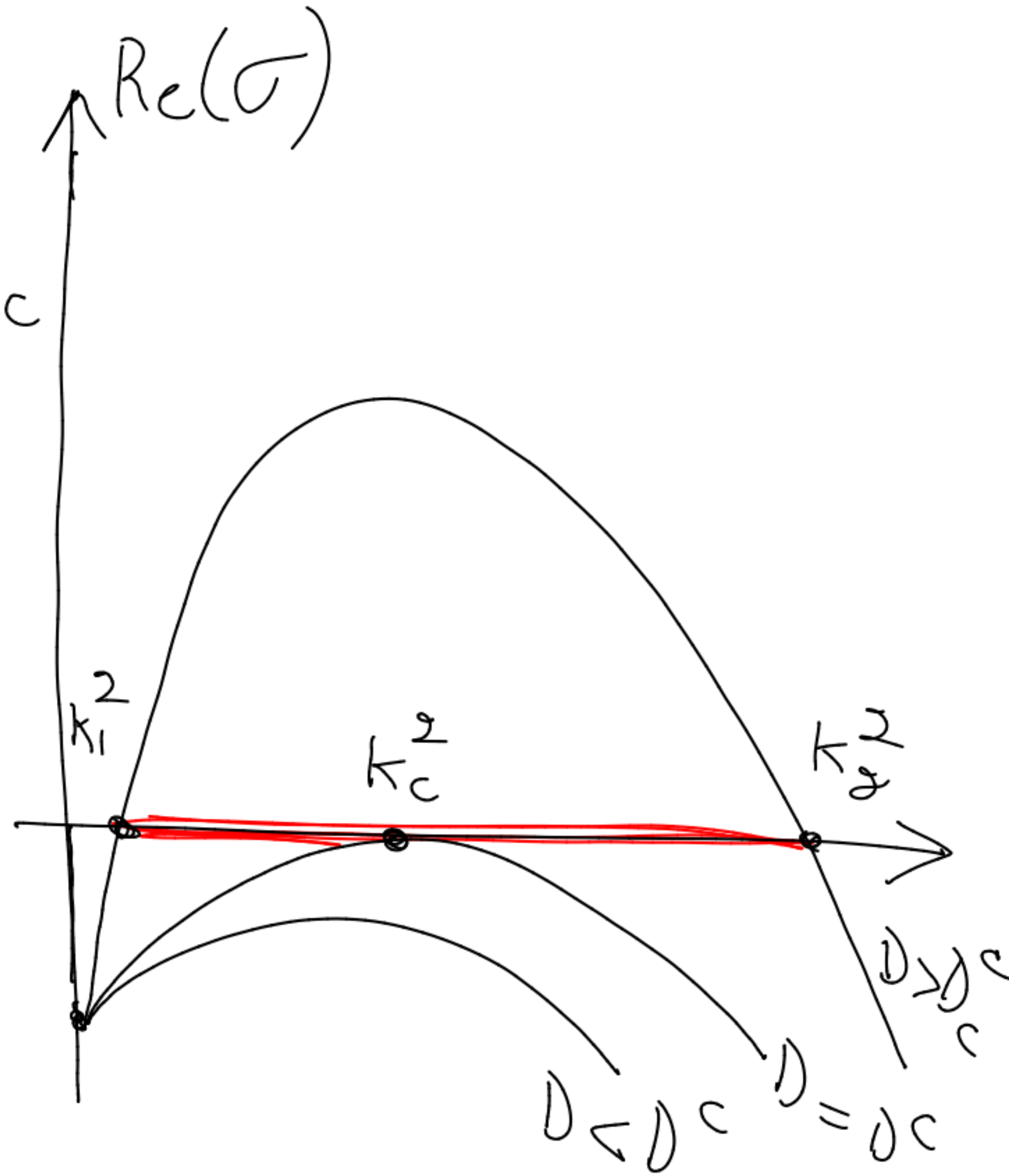
At bif. point, $h_{min} = 0 \iff |J| = \frac{1}{4D_u^c D_v^c}(aD_v^c - dD_u^c)^2$

$$k_c^2 = \frac{1}{2}\left(\frac{aD_v^c - dD_u^c}{D_u^c D_v^c}\right) = \frac{2\sqrt{D_u^c D_v^c}\sqrt{|J|}}{2D_u^c D_v^c} = \sqrt{\frac{|J|}{D_u^c D_v^c}} = \sqrt{\frac{bc-ad}{D_u^c D_v^c}}$$

Since $k^2 > 0$, then Turing instability occurs as long as $\frac{a}{D_u} > \frac{d}{D_v}$ and $h_{min} < 0$

Let $D^c = D_u^c D_v^c$


$h(k^2)$
$D<D^c$
$D=D^c$
$D>D^c$
(k^2)
k_1^2
k_c^2
k_2^2


$Re(\sigma)$
k_1^2
k_c^2
k_2^2
$D>D_c$
$D<D^c$
$D=D^c$

Example (Brusselator)

(Prigogine and Lefever 1974)

Reaction $\qquad\qquad\qquad\qquad$ Diffussion

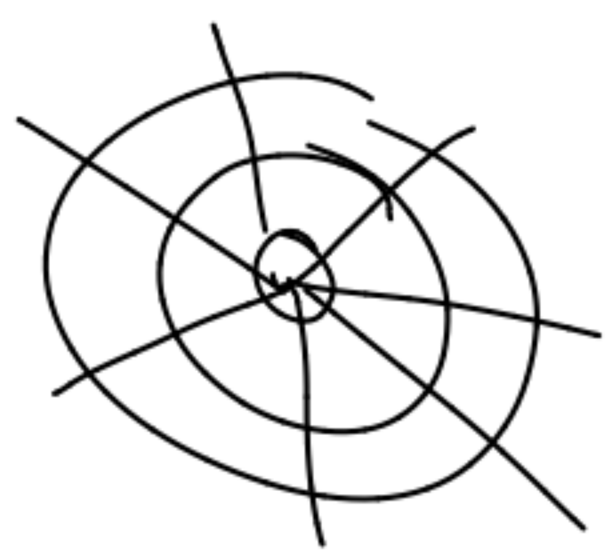
$$\frac{\partial X}{\partial t} = \underbrace{X^2 Y - (1+\beta)Y + \alpha}_{f(X,Y)} + D_X \frac{\partial^2 X}{\partial \xi^2}$$

$$\frac{\partial Y}{\partial t} = \underbrace{-X^2 Y + \beta X}_{g(X,Y)} + D_Y \frac{\partial^2 Y}{\partial \xi^2}$$

X, Y, α, β are chemical concentrations

$X = X(r, \theta)$

$Y = Y(r, \theta)$



$\xi = (r, \theta)$, $\quad 0 \le \xi \le R$