Brusselator

$$\frac{\partial X}{\partial t} = X^2 Y - (1+\beta)X + \alpha + D_X \frac{\partial^2 X}{\partial \xi^2}$$

$$\frac{\partial Y}{\partial t} = -X^2 Y + \beta X + D_Y \frac{\partial^2 Y}{\partial \xi^2}$$

Reactions:

$\alpha \xrightarrow{R_1} X$

$2X+Y \xrightarrow{R_2} 3X$

$\beta + X \xrightarrow{R_3} Y + d$

$X \xrightarrow{R_4} c$

$R_i \equiv$ Reaction rates

$\alpha, \beta \equiv$ original substances

$c, d \equiv$ final products

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Equilibrium Sol

$$X^2 Y - (1+\beta)X + \alpha + D_X \nabla^2 X \overset{set}{=} 0$$

$$-X^2 Y + \beta X + D_Y \nabla^2 Y = 0 \Rightarrow (\alpha, \beta/\alpha) = (X_E, Y_E)$$

Shift coord: $u = X - \alpha$

$v = Y - \beta/\alpha \Rightarrow \frac{\partial u}{\partial t} = \frac{\partial X}{\partial t}$, $\frac{\partial v}{\partial t} = \frac{\partial Y}{\partial t}$

$$\frac{\partial u}{\partial t} = (u+\alpha)^2\left(v+\frac{\beta}{\alpha}\right) - (1+\beta)(u+\alpha) + \alpha + D_u \nabla^2 u$$

$$\frac{\partial v}{\partial t} = -(u+\alpha)^2\left(v+\frac{\beta}{\alpha}\right) + \beta(u+\alpha) + D_v \nabla^2 v$$

$$\Downarrow$$

$$\frac{\partial u}{\partial t} = \frac{\beta}{\alpha}u^2 + 2\alpha uv + u^2 v + (\beta-1)u + \alpha^2 v + D_u \nabla^2 u$$

$$\frac{\partial v}{\partial t} = -\left(\frac{\beta}{\alpha}u^2 + 2\alpha uv + u^2 v\right) - \beta u - \alpha^2 v + D_v \nabla^2 v$$

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} \beta-1 & \alpha^2 \\ -\beta & -\alpha^2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} D_u & 0 \\ 0 & D_v \end{bmatrix}\begin{bmatrix} \nabla^2 u \\ \nabla^2 v \end{bmatrix}}_{\text{Linear Terms}} + \underbrace{\left(\frac{\beta}{\alpha}u^2 + 2\alpha uv + u^2 v\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{Nonlinear}}$$

New equilibrium: $(u_0 = 0, v_0 = 0)$

Perturbation $u = 0 + u$, $v = 0 + \hat{v}$, $\begin{bmatrix} u \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} e^{i\vec{k}\cdot\vec{x} + \sigma t} + c.c.$

Linearization about $(0,0)$

$$\nabla^2 \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = -k^2 \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}, \quad k^2 = |\vec{k}|^2$$

$$\begin{bmatrix} \frac{\partial \tilde{u}}{\partial t} \\ \frac{\partial \tilde{v}}{\partial t} \end{bmatrix} = \underbrace{\begin{bmatrix} \beta - 1 - k^2 D_u & \alpha^2 \\ -\beta & -\alpha^2 - k^2 D_v \end{bmatrix}}_{L} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \xrightarrow{\text{e-vals:}} \sigma^2 - tr(J)\sigma + det(J) -$$

$$tr(L) = -\alpha^2 - 1 + \beta - k^2(D_u + D_v)$$

$$det(L) = k^2\left[\alpha^2 D_u + (1-\beta) D_v\right] + k^4 D_u D_v + \alpha^2 \beta, = h(k^2)$$

$$= k^4 D_u D_v + k^2\left[\alpha^2 D_u + (1-\beta) D_v\right] + \alpha^2 (\beta)^{?} = h(k^2)$$

$$\mathrm{tr}(L) = 0 \iff \beta = 1 + \alpha^2 + k^2(D_u + D_v)$$

$$h'(k^2) = 2k^2 D_u D_v + \alpha^2 D_u + (1-\beta) D_v \overset{set}{=} 0$$

$$k^2 = \frac{1}{2D_uD_v}\left[(\beta-1)D_v - \alpha^2 D_u\right] \Rightarrow \boxed{k_c^2 = \frac{1}{2}\left(\frac{\beta-1}{D_u} - \frac{\alpha^2}{D_v}\right)}$$

$$h_{min} = h(k_c^2) \overset{algebra}{=} \dots = \alpha^2 - \frac{1}{4D_uD_v}\left[\alpha^2 D_u - (\beta-1)D_v\right]^2$$

At bif. point, $h_{min} = 0 \iff \alpha^2 = \frac{1}{4D_uD_v}\left[\alpha^2 D_u - (\beta-1)D_v\right]^2$

Solve for α^2: $\alpha^2 D_u - (\beta-1)D_v = \pm 2\alpha\sqrt{D_uD_v}$

Choose "$-$" since $D_u \ll D_v$ and $\beta > 1$

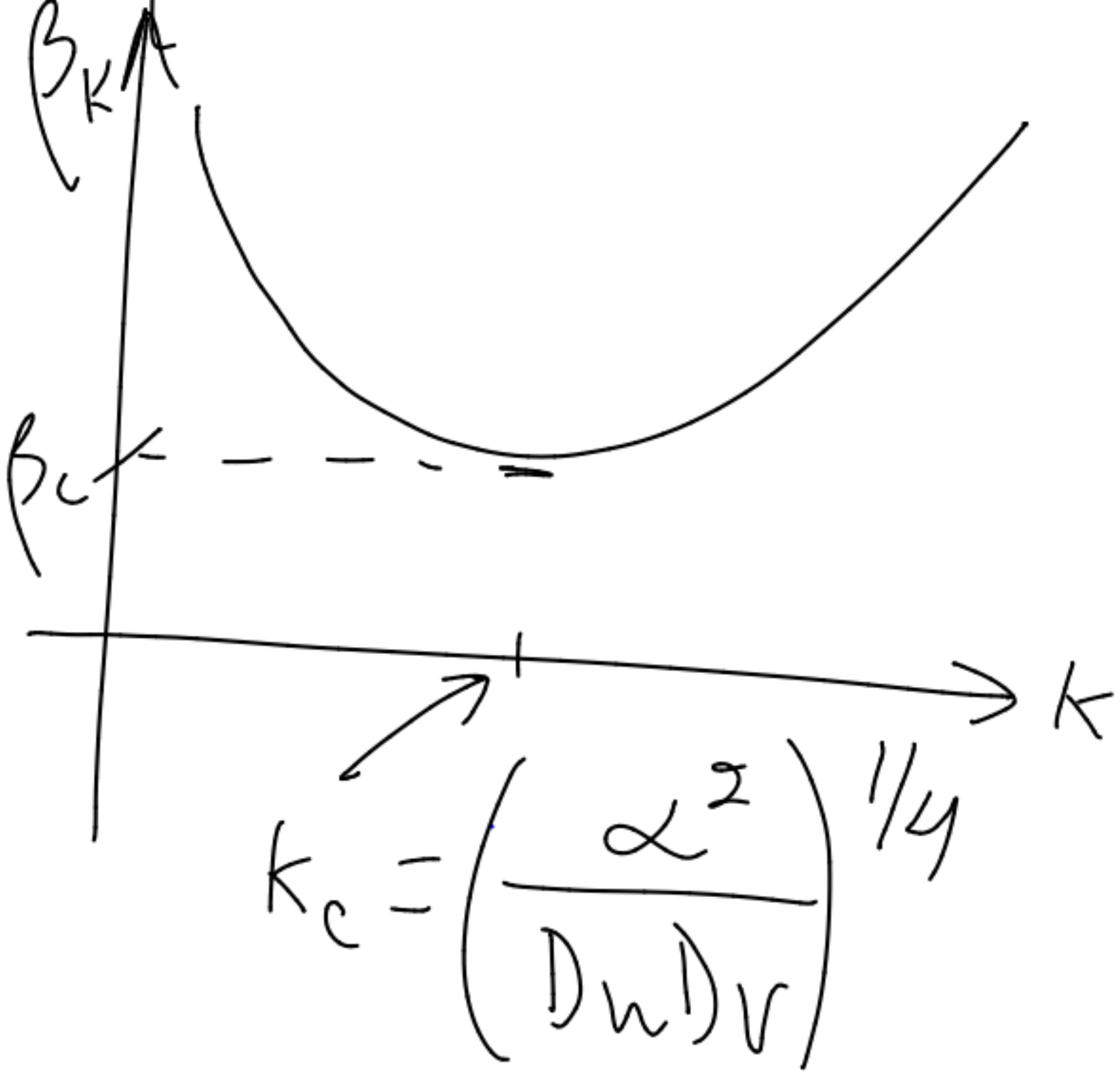
$$(\beta-1)\,D_v = \alpha^2 D_u + 2\alpha\sqrt{D_u D_v}$$

$$\beta - 1 = \alpha^2\frac{D_u}{D_v} + 2\alpha\sqrt{\frac{D_u}{D_v}}$$

$$\beta_c = 1 + \alpha^2\frac{D_u}{D_v} + 2\alpha\sqrt{\frac{D_u}{D_v}} \Rightarrow \boxed{\beta_c = \left(1 + \alpha\sqrt{\frac{D_u}{D_v}}\right)^2}$$

$$\Rightarrow k_c^2 = \frac{1}{2D_u}\left[\alpha^2\frac{D_u}{D_v} + 2\alpha\sqrt{\frac{D_u}{D_v}}\right] - \frac{1}{2D_v}\left(\alpha^2\right)$$

$$= \frac{\frac{1}{2}\alpha^2 D_u + \alpha\sqrt{D_u D_v} - \frac{1}{2}\alpha^2 D_u}{D_u D_v} = \frac{\alpha}{\sqrt{D_u D_v}} \Rightarrow \boxed{k_c = \left(\frac{\alpha^2}{D_u D_v}\right)^{1/4}}$$



$$k_c = \left(\frac{\alpha^2}{D_u D_v} \right)^{1/4}$$

$$\begin{aligned}\frac{\partial x}{\partial t} &= \overbrace{X^2Y-(1+\beta)X+\alpha}^{f_1(x,y)}+D_X\nabla^2X\\ \frac{\partial y}{\partial t} &= \underbrace{-X^2Y+\overset{\beta}{\beta}X}_{f_2}+D_Y\nabla^2Y\end{aligned}\quad , \quad X_E=\alpha,\ Y_E=\frac{\beta}{\alpha}$$

Equil.

S1

$$J=\begin{bmatrix}2\overset{x}{\textcircled{u}}\overset{y}{\textcircled{v}}-(1+\beta) & u^2\\ \beta-2uv & -u^2\end{bmatrix}\Bigg|_{(X_E,Y_E)}=\begin{bmatrix}\overset{a}{\textcircled{\beta-1}} & \textcircled{\alpha^2}\,{-b}\\ c\ \textcircled{-\beta} & \textcircled{-\alpha^2}\,{-d}\end{bmatrix}$$

$$|J|=\ldots=\alpha^2$$

$$\Rightarrow h_{min}=\alpha^2-\frac{1}{4D_uD_v}\left[\alpha^2D_u-(\beta-1)D_v\right]^2\overset{set}{=}0$$

$$\Rightarrow \alpha^2D_u-(\beta-1)D_v=\pm 2\alpha\sqrt{D_uD_v}\Rightarrow \beta_{min}=1+2\alpha\sqrt{\frac{D_u}{D_v}}+\alpha^2\frac{D_u}{D_v}$$

$$k_c=\left(\frac{\alpha}{D_uD_v}\right)^{1/4}$$