M635 Pattern Formation

Midterm Exam - Spring 2019

Instructions: Show all work.

Name: ______________________________________

I, ________________________, pledge that this exam is completely my own work, and that I did not take, borrow or steal any portions from any other person. Any and all references I used are clearly cited in my solutions. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

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Signature

1. (25 Pts) The Kuramoto-Sivashinsky Equation

Consider the Kuramoto-Sivashinsky model of flame instability:

$$\frac{\partial u}{\partial t} = \varepsilon u - (1+\nabla^2)^2 u - \eta_1 (\nabla u)^2 - \eta_2 u^3, \tag{1}$$

where $u = u(\mathbf{x}, t)$ represents the perturbation of a planar flame front in the direction of propagation, ε measures the strength of the perturbation force, η_1 is a parameter associated with growth in the direction normal to the burner, and $\eta_2 u^3$ is a term that was added to help stabilize the numerical integration. This model is derived by making a series of simplifying assumptions on a pair of diffusion equations (for a single chemical species and temperature $u(\mathbf{x}, t)$) coupled to fluid equations.

Study the stability properties of the homogeneous trivial solution $u_0 = 0$ by performing the following tasks:

(a) Linearize (1) about u_0: Set $u = u_0 + \tilde{u}$ and write the linearized equation for $\tilde{u}$.

(b) Let $\tilde{u}(\vec{x}, t) = \hat{u} e^{\sigma t + i\vec{k}\cdot\vec{x}}$, where $\vec{x} = (x, y)$. Substitute and solve the resulting eigenvalue problem for σ.

(c) Compute marginal stability curves $\sigma = 0$ and study the behavior of the perturbation for various values of n in the (ε, R) plane. Assume $|\vec{k}|^2 = (\alpha_{nm}/R)^2$, where R is the radius of a circular burner.

(d) Use MATLAB or equivalent software to plot a few marginal stability curves ε_{nm} as functions of radius R. Also, plot a few representative examples of Fourier-Bessel modes.

2. (20 Pts) Restrictions Imposed by Symmetry

Consider the system

$$\begin{aligned} \frac{dx}{dt} &= f(x, y, z) \\ \frac{dy}{dt} &= g(x, y, z) \\ \frac{dz}{dt} &= h(x, y, z). \end{aligned} \tag{2}$$

under the symmetries:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y \\ z \\ x \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ z \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix},$$

Show that these symmetries imply

$$\begin{aligned} h(y,z,x) &= g(z,x,y) = f(x,y,z) \\ f(x,y,z) &= f(x,z,y) \\ f(x,y,z) &= -f(-x,-y,-z) \end{aligned}$$

and hence write down the series expansion for $f(x,y,z)$, $g(x,y,z)$ and $h(x,y,z)$ for small $|x|$, $|y|$ and $|z|$ to quadratic order.

3. (25 Pts) Steady-State Bifurcation in a Triangle

Let $\Gamma = \mathbf{D}_3 = \langle \gamma,\ \kappa \rangle$ act on $\mathbf{C} = \mathbf{R}^2$ by its standard or natural action:

$$\begin{aligned} \gamma \cdot z &= e^{\frac{2\pi}{3}i}\, z \\ \kappa \cdot z &= \bar{z} \end{aligned} \tag{3}$$

(a) Let $p : \mathbf{R}^2 \to \mathbf{R}$ be a polynomial of the form

$$p(z) = p(z\bar{z}, z^3 + \bar{z}^3).$$

Show that p is invariant under the standard action given by Eq. (3).

(b) Consider a differential equation of the following form:

$$\frac{dz}{dt} = f(z) = p(z\bar{z}, z^3 + \bar{z}^3, \mu)\, z + q(z\bar{z}, z^3 + \bar{z}^3, \mu)\, \bar{z}^2, \tag{4}$$

where p and q are $\mathbf{D}_3$-invariant polynomials. Show that Eq. (4) is equivariant under the standard action given by Eq. (3).

(c) Sketch the Lattice of Isotropy Subgroups of $\Gamma = \mathbf{D}_3$.

(d) Study the stability properties of each bifurcating pattern. Hints: (i) Consider the isotypic decomposition: $\mathbf{C} = \mathbf{R}^2 = \mathbf{R} \oplus \mathbf{R}\{i\}$. (ii) The Jacobian in complex coordinates can be computed as $(df)v = f_z v + f_{\bar{z}} \bar{v}$.

4. (30 Pts) Steady-State Bifurcation in a Hexagon

(a) Find all the absolutely irreducible representations of $\mathbf{D}_6$, the symmetry group of a hexagon.

(b) Work out all the possible solutions that are guaranteed at a steady bifurcation with $\mathbf{D}_6$ symmetry under one or other of these representations.

(c) Draw examples of the eigenmodes in an appropriate hexagonal box, and derive the relevant normal form equations.