PATTERN FORMATION

A Group Theoretical Approach

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Outline

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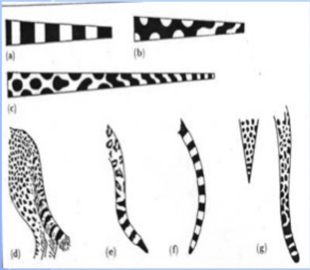
1. Overview
2. PDE Approach
   - Turing Instability
   - Combustion
   - Analysis
3. Symmetry Approach
   - Group Theory
   - Steady-State Bifurcation with $\mathbf{D}_4$-Symmetry
   - Lattice Patterns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(e)
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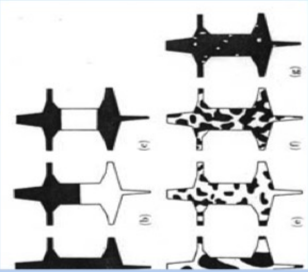

Turing patterns in Nature

Papilio dardanus


nature
INTERNATIONAL WEEKLY JOURNAL OF SCIENCE
Volume 376 No. 6543 31 August 1995
Turing patterns come to life
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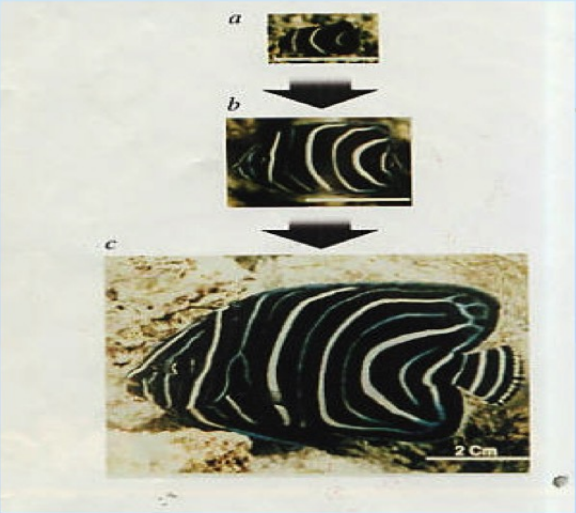
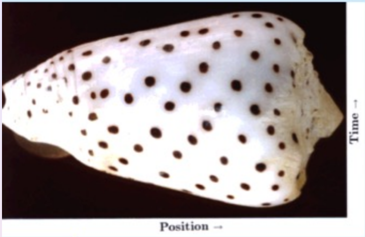
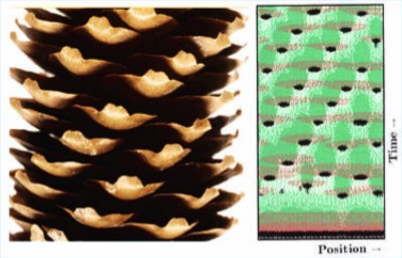
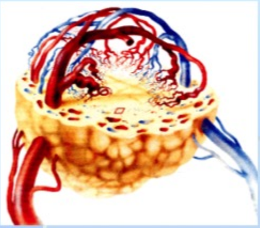
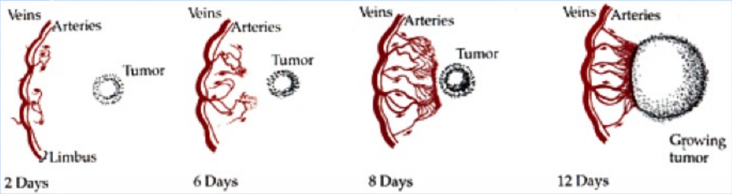
Photographs of the juvenile of P. Semicirculatus




Time →
Position →

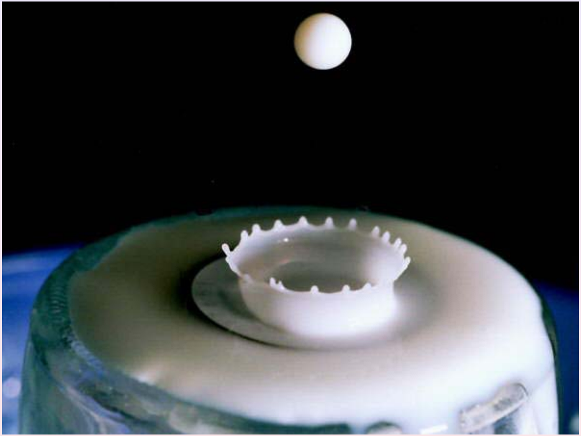

Time →
Position →



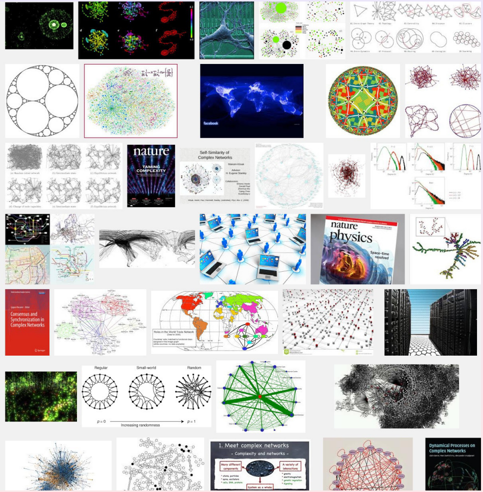


Tumor growth and vascularisation steps. The arteries (red network) bring Oxygen and nutrients to the tumor

Milk Splash



Complex Networks



Alan Turing (1952)

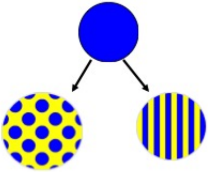
$$\frac{\partial u}{\partial t} = f(u,v) + D_u \frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial v}{\partial t} = g(u,v) + D_v \frac{\partial^2 v}{\partial x^2}, \qquad (1)$$

where $u = u(x,t)$, $v = v(x,t)$, x represents space and t is time.

Novelty: Stable system + "Stabilizing" diffusion = Instability

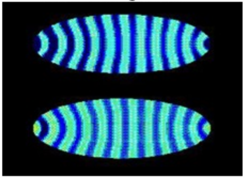
A.M. Turing. The Chemical Basis of Morphogenesis. Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences. Vol. 237, no. 641. (Aug. 14, 1952), pp. 37-72.

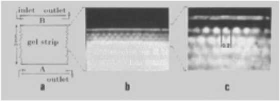
Turing Patterns



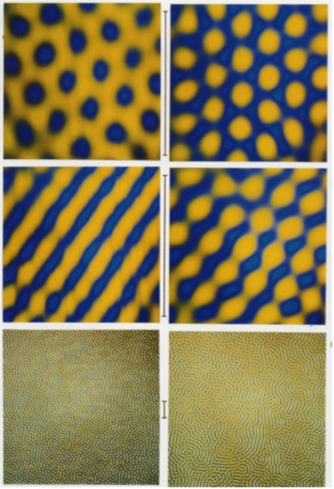
- Turing proposal for "morphogenesis" (1952)
- "selective diffusion" in reactions with feedback
- requires diffusivity of feedback species to be reduced compared to other reactants
- recently observed in experiments
- not clear that this underlies embryo development

A. Hunding, 2000





Castets *et al.* Phys Rev. Lett 1990



Ouyang and Swinney
Chaos 1991

CDIMA reaction
Turing Patterns

spots and stripes:
depending on
Experimental
Conditions

Model reaction kinetics

- Gierer-Meinhardt (1972) – Activator-Inhibitor

$$\frac{\partial u}{\partial t} = \nabla^2 u + \gamma\left(a - b\,u + \frac{u^2}{v(1+k\,u^2)}\right),$$

$$\frac{\partial v}{\partial t} = d\,\nabla^2 v + \gamma\,(u^2 - v)$$

- Thomas (1975) – Substrate-Inhibition

$$\frac{\partial u}{\partial t} = \nabla^2 u + \gamma\,(a - u - h(u,v)),$$

$$\frac{\partial v}{\partial t} = d\,\nabla^2 v + \gamma\,(\alpha(b - v) - h(u,v)), \quad \text{with} \quad h(u,v) = \frac{\rho\,u\,v}{1 + u + K\,u^2}.$$

- Schnakenberg (1979) – Chemically derived

$$\frac{\partial u}{\partial t} = \nabla^2 u + \gamma\,(a - u + u^2 v),$$

$$\frac{\partial v}{\partial t} = d\,\nabla^2 v + \gamma\,(b - u^2 v)$$

Robustness of solutions

- Turing patterns only develop in particular points of the parameter space (Turing space), tight control of the parameters is required for pattern formation
- Pattern selection, especially when multiple unstable modes exist, is strongly dependent on the initial conditions (Acuri and Murray, 1986)
- **Domain growth enhances robustness in pattern selection and transition!**

RDEs on growing domains

Consider the model equations: (Crampin, *et. al.* 2002; Madzvamuse, *et, al.*, 2003,2005, 2005)

$$\frac{\partial u}{\partial t} + \nabla.(\mathbf{a}\, u) = D_u \nabla^2 u + F(u,v)$$

$$\frac{\partial v}{\partial t} + \nabla.(\mathbf{a}\, v) = D_v \nabla^2 v + G(u,v), \text{ in } \Omega(t)$$

Boundary conditions:

Boundary conditions can be of Dirichlet type (u and/or v given on the boundary) or of (homogeneous) Neumann type which describe zero-flux of u (or v) out of the boundary.

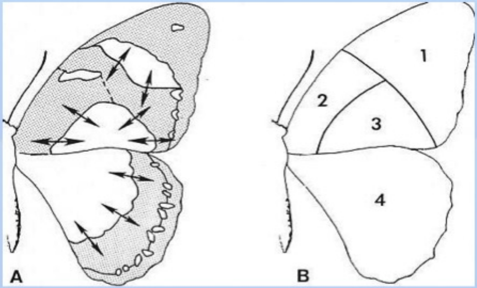
Initial conditions:

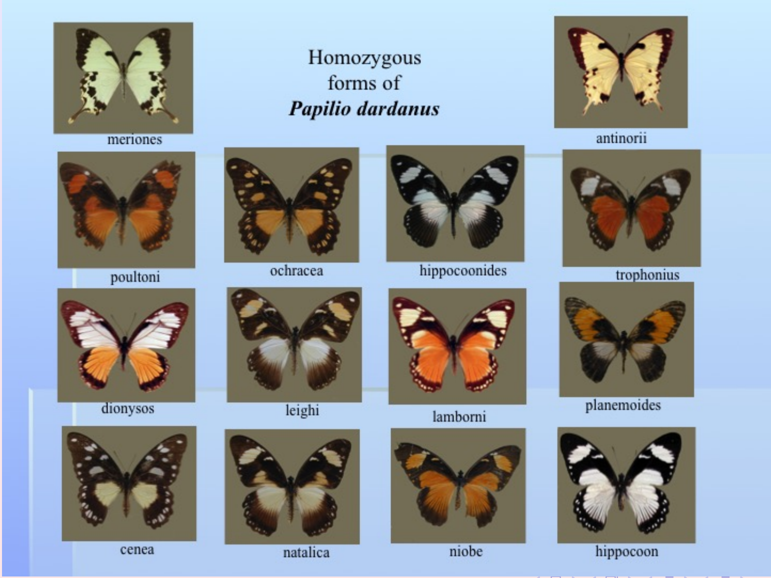
Initial conditions are taken as small perturbations around the homogeneous steady state if it exits.

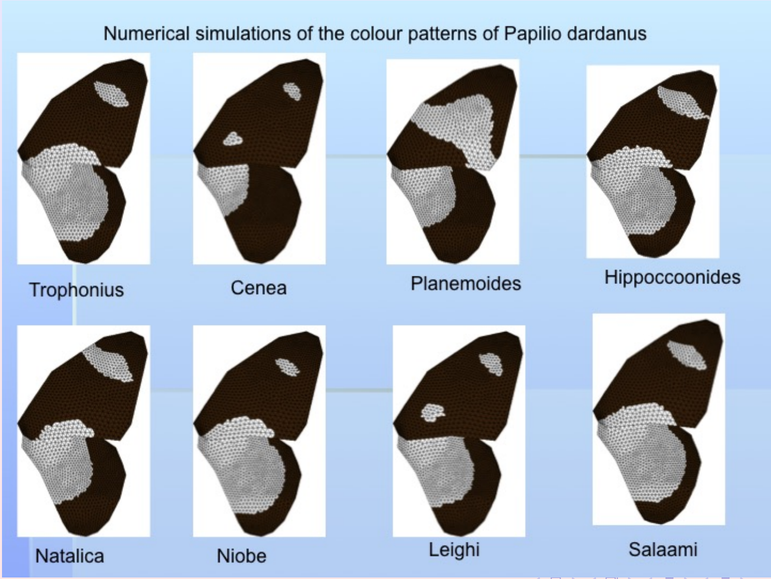
Nymphalid ground plan

(Schwanwitsch, 1924; Suffert, 1927; Nijhout,1991)

- Complicated patterns can be understood as a composite of relatively small number of element patterns




Homozygous
forms of
Papilio dardanus
meriones
antinorii
poultoni
ochracea
hippocoonides
trophonius
dionysos
leighi
lamborni
planemoides
cenea
natalica
niobe
hippocoon


Numerical simulations of the colour patterns of Papilio dardanus
Trophonius
Cenea
Planemoides
Hippoccoonides
Natalica
Niobe
Leighi
Salaami

Exact patterning of the butterfly

Papilio Dardanus is compared with the results of numerical computation of a reaction-diffusion system-the Gierer-Meinhardt model

Cenea

Trophonius

Planemoides

Hippoccoonides

Linear Stability

$$\frac{\partial u}{\partial t} = f(u,v) + D_u \frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial v}{\partial t} = g(u,v) + D_v \frac{\partial^2 v}{\partial x^2},$$

rewritten as

$$\frac{\partial \mathbf{w}}{\partial t} = F(\mathbf{w},\mu) + D\nabla^2 \mathbf{w}$$

where $\mathbf{w} = (u, v)$.

Let $\mathbf{w_0} = (u_0, v_0)$ be a homogeneous sol. and let $\mathbf{w} = \mathbf{w_0} + \delta\mathbf{w}$, where

$$\delta\mathbf{w} = \sum_j c_j e^{\lambda_j t} e^{-ik_j \cdot \mathbf{x}}.$$

Substituting into linearized system about $\mathbf{w_0} = (u_0, v_0)$ yields:

$$(J - Dk_j^2 - \lambda_j)\mathbf{w} = 0, \qquad (2)$$

where $k_j^2 = \vec{k}_j \cdot \vec{k}_j$ and

$$J = \begin{pmatrix} \partial_u f & \partial_v f \\ \partial_u g & \partial_v g \end{pmatrix}_{(u_0,v_0)}$$

Dispersion relation

Solving Eq. (2) yields:

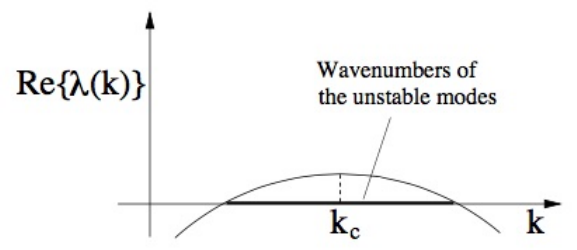
$$\lambda^2+((D_u+D_v)k^2-f_u-g_v)\lambda+D_uD_vk^4-k^2(D_vf_u+D_ug_v)+f_ug_v-f_vg_u=0$$

Pattern Selection Mechanism:

$\lambda(k)$ predicts the growing wave modes: $We^{i\vec{k}\cdot\vec{r}}e^{\lambda(k)t}$.

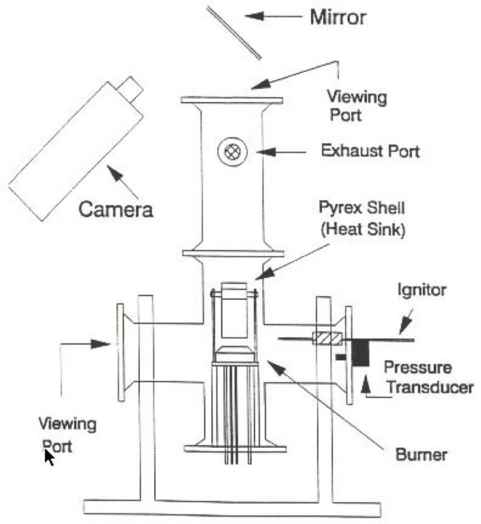
Wave numbers k with $Re\{\lambda(k)\}>0$ grow exponentially until the nonlinearities in the reaction kinetics bound this growth.

Critical Wave Number: Solve $\lambda(\mathbf{k_c}) = 0$ for $\mathbf{k_c}$



where

$$\mathbf{k_c^2} = \frac{D_v f_u + D_u g_v}{2 D_u D_v}$$


Mirror
Viewing
Port
Exhaust Port
Camera
Pyrex Shell
(Heat Sink)
Ignitor
Pressure
Transducer
Viewing
Port
Burner

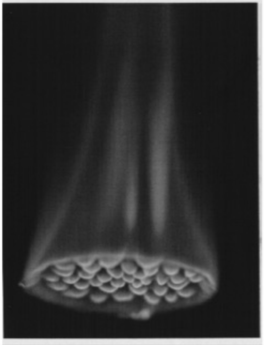
"Turing Patterns" in flames

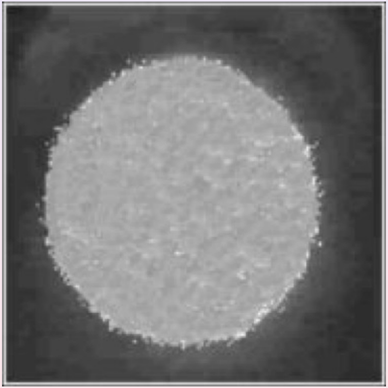
"thermodiffusive instability"

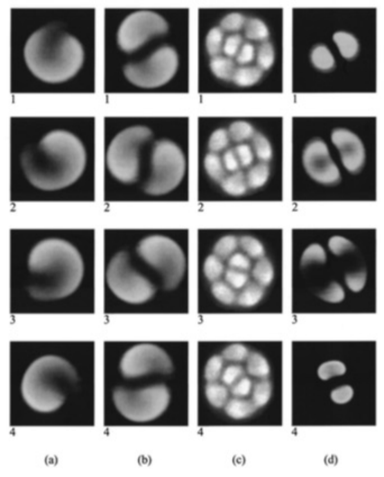
- first observed in Leeds

(Smithells & Ingle 1892)

requires thermal diffusivity $<$ mass diffusivity






1
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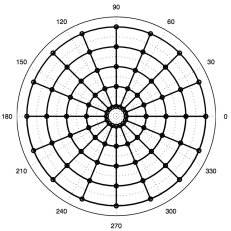
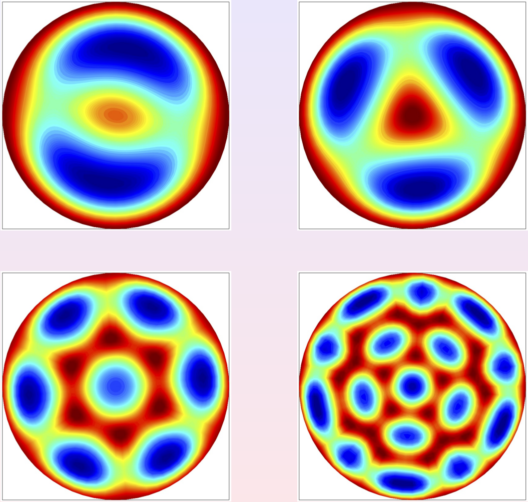
4
4
4
4
(a)
(b)
(c)
(d)

Kuramoto-Shivashinky Model

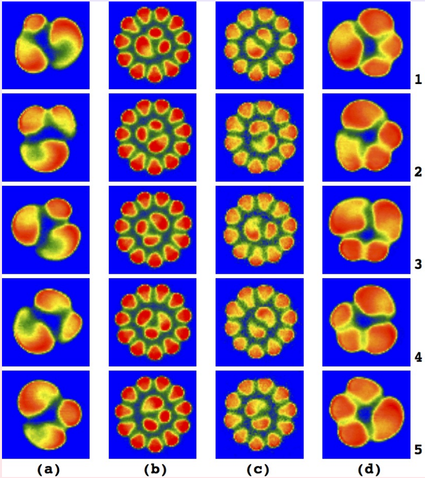
$$\frac{\partial u}{\partial t} = \varepsilon u - (1 + \nabla^2)^2 u - \eta_1 (\nabla u)^2 - \eta_2 u^3,$$

where
$u = u(\mathbf{x}, t) \equiv$ perturbation of a planar front with strength ε,
$\eta_1 \equiv$ parameter associated with growth in the direction normal to the circular domain (burner),
$\eta_2 u^3$ is added to stabilize the numerical integration.





Hopping Patterns



Analysis: Proper Orthogonal Decomposition (POD)

Given a data set $\{u(\mathbf{x}, t)\}$ the POD extracts orthogonal basis $\{a_k(t), \Phi_k(\mathbf{x})\}$ such that the reconstruction

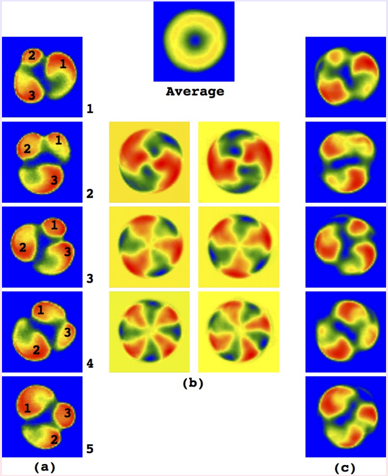
$$u_{approx}(\mathbf{x}, t) = \tilde{u} + \sum_{k=1}^{M} a_k(t)\,\Phi_k(\mathbf{x}),$$

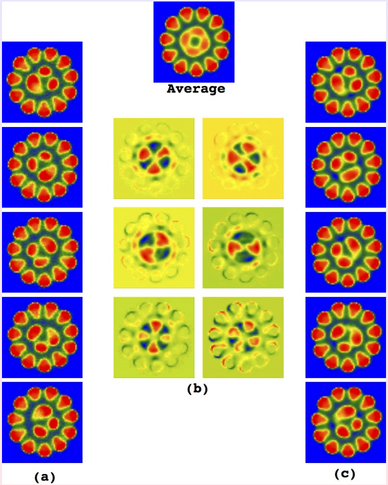
is optimal in the sense that the average least squares truncation error

$$e_n = \langle \| u((\mathbf{x}, t_i) - u_{approx}(\mathbf{x}, t) \|_2 \rangle$$

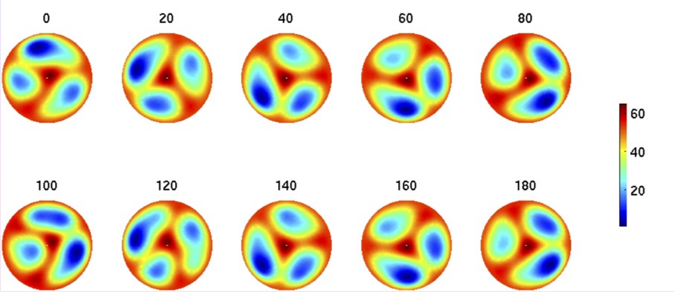
where $\tilde{u}$ is the time-average of the data set $u(\mathbf{x}, t)$.

Note: POD is also known as Principal Component Analysis.


Average
2 1 3
1
2 1 3
2
1 2 3
3
1 3 2
4
1 3 2
5
(a)
(b)
(c)


Average
(a)
(b)
(c)

3-Cell Hopping Pattern

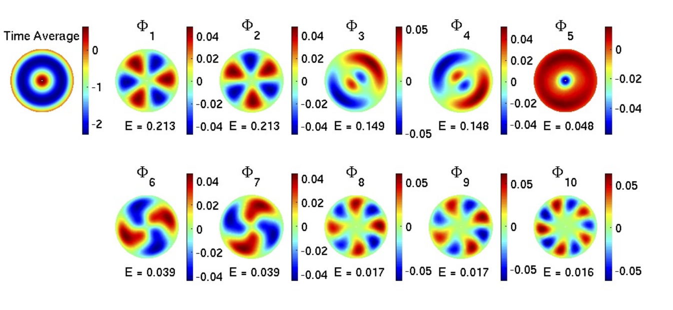


A. Palacios, P. Blomgren and S. Gasner. Bifurcation analysis of hopping behavior in cellular pattern-forming systems. Int. J. Bif. and Chaos, **17**, no. 2, (2007) 509-520.

Analysis

$$u(\mathbf{x}, t) = \tilde{u} + z_2(t)\Psi_{21}(\mathbf{x}) + z_3(t)\Psi_{31}(\mathbf{x}) + z_4(t)\Psi_{41}(\mathbf{x}) + c.c.$$

where $\tilde{u}$ is the time-average of the data set $u(\mathbf{x}, t)$.



Idealization: Γ-Equivariant System of ODEs

$$\frac{dz}{dt} = f(z, \mu),$$

where $\Gamma = \mathbf{O}(\mathbf{2})$ represents the circular symmetry of the burner, $z = (z_2, z_3, z_4) \in \mathbf{C}^3$, and $\mu \in \mathbf{R}^3$ are vectors of parameters.

Assume:

- $z = 0$ to be “trivial solution” (planar front): $f(0, \mu) = 0$.
- Bifurcation occurs at $\mu = (0, 0, 0)$, so that $V = ker(Df) \neq 0$.
- $L = (Df)_{0,(0,0,0)}$ has three zero eigenvalues.
- $V = V_2 \oplus V_3 \oplus V_4$, where $V_k = span\{Re\{\Psi_{k1}\}, Im\{\Psi_{k1}\}\}$.

Symmetry-Breaking Bifurcations

Under these assumptions, at $\mu = (0, 0, 0)$, the $z = 0$ uniform solution loses stability and three $\mathbf{O(2)}$ symmetry-breaking branches of steady-states modes interact with each other. The action of Γ on C^3 is generated by:

Group Action

$$\begin{aligned} \theta \cdot (z_2, z_3, z_4) &= (e^{2\theta i} z_2, e^{3\theta i} z_3, e^{4\theta i} z_4), & \theta \in SO(2), \\ \kappa \cdot (z_2, z_3, z_4) &= (\bar{z}_2, \bar{z}_3, \bar{z}_4), & \kappa = \text{flip}. \end{aligned}$$

Invariant Theory

Painful derivations . . . yield the amplitude equations (ODEs)

$$
\begin{aligned}
\dot{z}_2 &= \bar{z}_2 z_4 + \alpha_2 z_3^2 \bar{z}_4 + z_2(\mu_2 + e_{22}|z_2|^2 + e_{23}|z_3|^2 + e_{24}|z_4|^2) \\
\dot{z}_3 &= \alpha_3 z_2 \bar{z}_3 z_4 + z_3(\mu_3 + e_{32}|z_2|^2 + e_{33}|z_3|^2 + e_{34}|z_4|^2) \\
\dot{z}_4 &= \pm z_2^2 + \alpha_4 z_3^2 \bar{z}_2 + z_4(\mu_4 + e_{42}|z_2|^2 + e_{43}|z_3|^2 + e_{44}|z_4|^2),
\end{aligned}
$$

where α_2, α_3, and α_4 are real-valued constants.

Abstract Groups

A group Γ is a set $\{\gamma_1, \gamma_2, \ldots\}$ that satisfies:

- Closure: $\gamma \times \Gamma \longrightarrow \Gamma$ or $\gamma_1\gamma_2 = \gamma_3 \in \Gamma$
- Associativity: $(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$
- Identity: $\exists e \in \Gamma$ st. $\gamma e = e\gamma = \gamma$
- Inverses: $\exists \gamma^{-1} \in \Gamma$ st. $\gamma\gamma^{-1} = \gamma^{-1}\gamma = e$.

Examples:

- $Z_N = \{0, 1, \ldots, N-1\}$
- $\mathbf{S}^1$ = Circle Group = $\{z \in \mathbf{C} : |z| = 1\}$
- $GL(n)$: General linear group of real invertible $n \times n$ matrices.
- $\mathbf{O}(n)$: Set of $n \times n$ orthogonal matrices, $A^T = A^{-1}$.

Table 3.1. *A group table*

	e	γ_1	γ_2	$\cdots$	γ_n
e	e	γ_1	γ_2	$\cdots$	γ_n
γ_1	γ_1	γ_1^2	$\gamma_1\gamma_2$	$\cdots$	$\gamma_1\gamma_n$
γ_2	γ_2	$\gamma_2\gamma_1$	γ_2^2	$\cdots$	$\gamma_2\gamma_n$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
γ_n	γ_n	$\gamma_n\gamma_1$	$\gamma_n\gamma_2$	$\cdots$	γ_n^2

Table 3.2. *The group table for $\mathcal{D}_3$, the symmetry group of an equilateral triangle*

	e	ρ	ρ^2	m	$m\rho$	$m\rho^2$
e	e	ρ	ρ^2	m	$m\rho$	$m\rho^2$
ρ	ρ	ρ^2	e	$m\rho^2$	m	$m\rho$
ρ^2	ρ^2	e	ρ	$m\rho$	$m\rho^2$	m
m	m	$m\rho$	$m\rho^2$	e	ρ	ρ^2
$m\rho$	$m\rho$	$m\rho^2$	m	ρ^2	e	ρ
$m\rho^2$	$m\rho^2$	m	$m\rho$	ρ	ρ^2	e

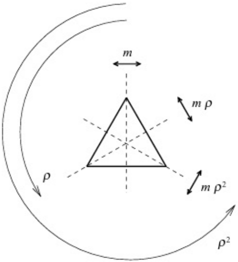


Fig. 3.1. The elements of $\mathcal{D}_3$.

Subgroups

A subgroup $H \subseteq \Gamma$ is a subset that forms a group under the same group operation.

Example:

- $\mathbf{Z}_N \subset \mathbf{D}_N \quad$ or $\quad \mathbf{D}_N \subset \mathbf{O}(n)$
- Let $\Gamma = \mathbf{D}_3$. Then $H = \mathbf{Z}_2(\kappa) = \{e, m\} \subset \mathbf{D}_3$.

Normal Subgroups

H is a normal subgroup of Γ if

$$\gamma h \gamma^{-1} \in H, \quad \forall \gamma \in \Gamma, \quad \forall h \in H.$$

Representations of Groups

A representation of a finite group or compact Lie group, over a field, F, is a homomorphism $\rho : \Gamma \longrightarrow GL(n, F)$, i.e., $\rho(\gamma) = M_\gamma$.

The degree or dimension of the representation is n.

Example 3.21 *The group $\mathcal{D}_3$ has the representation*

$$M_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_m = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$M_\rho = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad M_{m\rho} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

$$M_{\rho^2} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad M_{m\rho^2} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

Γ-Invariant Subspaces

A subspace $W \subset V$ is Γ-invariant if $\theta(\gamma)w \in W$, $\forall \gamma \in \Gamma$ and $\forall w \in W$.
e.g.) $W = \{0\}$ and $W = V$ are always Γ-invariant.

Irreducible Representations

A representation or action of Γ is irreducible if the only Γ-invariant subspaces are $\{0\}$ and V.
e.g.) Trivial representation: $\rho_\gamma(x) = x$.
e.g.) $\Gamma = \mathbf{S}^1$ acting on $\mathbf{C}$: $\rho_\theta^k \cdot z = e^{ik\theta} z$.

Absolutely Irreducible Representations

An action or representation of Γ is said to be absolutely irreducible if the only linear mappings that commute with the action of Γ on V are scalar multiples of the identity.
e.g.) All 1D representations are abs. irrep, as is the natural rep. of $\mathbf{D}_3$.

Example 3.26 *In the standard action of $SO(2)$ on $\mathbb{R}^2$, a rotation through an angle, θ, is represented by the rotation matrix*

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{3.90}$$

The only $SO(2)$-invariant subspaces are the origin and the whole of $\mathbb{R}^2$, so this action is irreducible. However, it is not absolutely irreducible since each matrix R_θ commutes with every other rotation matrix $R_\phi \in SO(2)$:

$$R_\theta R_\phi = R_\phi R_\theta = R_{\theta+\phi}. \tag{3.91}$$

Theorem 3.1 (Orthogonality theorem for matrix representations) *Let the sets of matrices M_γ^p and M_γ^q belong to two unitary representations of a finite group Γ, where p and q label the representations, so that the representations are identical if $p = q$ and inequivalent if $p \neq q$. Then*

$$\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\overline{(M_\gamma^p)}_{jk}(M_\gamma^q)_{st} = \frac{1}{n_p}\delta_{pq}\delta_{js}\delta_{kt}, \tag{3.102}$$

where n_p is the dimension of M^p.

$$(M_\gamma)_{11} = \begin{pmatrix} (M_{\gamma_1})_{11} \\ (M_{\gamma_2})_{11} \\ (M_{\gamma_3})_{11} \\ \cdots \\ (M_{\gamma_n})_{11} \end{pmatrix}$$

Example 3.27 *Let us consider the natural and identity representations of $\mathcal{D}_3$. Both these representations are real, so there is no need to take the complex conjugate of a vector when forming the dot product. Taking the natural representation first, the vectors $M^a(\Gamma)_{ij}$ are given by*

$$(M_\Gamma^{\mathrm{n}})_{11} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, (M_\Gamma^{\mathrm{n}})_{21} = \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix},$$

$$(M_\Gamma^{\mathrm{n}})_{12} = \begin{pmatrix} 0 \\ -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}, (M_\Gamma^{\mathrm{n}})_{22} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \tag{3.104}$$

where the superscript n *denotes the natural representation, and the rows of the vectors range over the group elements in the order $\{e, \rho, \rho^2, m, m\rho, m\rho^2\}$. Any*

Characters

The **character** $\xi(M)$ of an $n \times n$ matrix M is its trace:

$$\xi(M) = \sum_{i=1}^{n} M_{ii}.$$

Example 3.28 *The characters of the natural representation of $\mathcal{D}_3$ are*

$$\chi_e = 2, \ \chi_\rho = -1, \ \chi_{\rho^2} = -1, \ \chi_m = 0, \ \chi_{m\rho} = 0, \ \chi_{m\rho^2} = 0. \tag{3.109}$$

The identity has character 2 as this is a two-dimensional irrep. All elements in the conjugacy class $\{m, m\rho, m\rho^2\}$ have character 0, while the two conjugate elements ρ and ρ^2 have character -1.

Theorem 3.2 *The number of inequivalent irreducible representations of a finite group* Γ *is equal to the number of conjugacy classes of* Γ.

This means that character tables are always square as shown in Table 3.3.

Theorem 3.3 *The sum of the squares of the dimensions* d_i *of the* n *inequivalent irreducible representations of a finite group* Γ, *is equal to the order,* $|\Gamma|$, *of* Γ:

$$\sum_{i=1}^{n} d_i^2 = |\Gamma|. \tag{3.112}$$

The proofs of Theorems 3.2 and 3.3 can be found in Cornwell (1984).

Theorem 3.4 (Orthogonality theorems for characters) *For a finite group,* Γ, *the characters satisfy*

$$\sum_{p=1}^{n} \overline{\chi^p(\mathcal{C}S_i)}\chi^p(\mathcal{C}S_j)N_i = |\Gamma|\delta_{ij}, \tag{3.113}$$

Example 3.29 *$\mathcal{D}_3$ has three conjugacy classes, so by Theorem 3.2 it must have three inequivalent irreps. By Theorem 3.3 the sum of the squares of the dimensions of the irreps must equal the order of the group, which is 6. That means that it must have two one-dimensional irreps and one two-dimensional irrep, since $1^2 + 1^2 + 2^2 = 6$, and there is no other way to add three square numbers to get 6. We already know about two of the irreps: the identity and natural representations, so we just need to find one more one-dimensional irrep, which turns out to be*

$$M_e^a = M_\rho^a = M_{\rho^2}^a = 1, \quad M_m^a = M_{m\rho}^a = M_{m\rho^2}^a = -1, \tag{3.115}$$

where a labels the irrep. In this irrep rotations and the identity are represented by $+1$ and reflections by -1, values equal to the determinant of the corresponding matrix in the natural representation. There is always a one-dimensional irrep of this kind for $\mathcal{D}_n$.

Table 3.4. *The character table for $\mathcal{D}_3$. The labels i and n denote the identity and natural representations respectively, and the label a denotes the irrep defined in equation (3.115)*

Irrep	$\{e\}$	$\{\rho, \rho^2\}$	$\{m, m\rho, m\rho^2\}$
i	1	1	1
a	1	1	-1
n	2	-1	0

Let Γ be a compact Lie group acting on a vector space V. $\exists$ Γ-irreducible subspaces U_i of V st.

$$V = U_1 \oplus U_2 \oplus \ldots \oplus U_m$$

Isotypic Decomposition

This decomposition is not unique. So let

$$\begin{aligned} W_1 &= U_1 \oplus U_2 \oplus \ldots \oplus U_{n1} \\ &\vdots \\ W_k &= U_1 \oplus U_2 \oplus \ldots \oplus U_{nk}. \end{aligned}$$

Then

$$V = W_1 \oplus W_2 \oplus \ldots \oplus W_k, \quad k < m,$$

is uniques and it is called the Isotypic Decomposition of V.

Example 3.30 *Let $SO(2)$ act by matrix multiplication on the space, V, of 2×2 real matrices, such that a rotation, ρ, through an angle θ is given by*

$$\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.124}$$

Any 2×2 real matrix can be expressed as the sum of two other matrices in the manner

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}, \tag{3.125}$$

and so we can write $V = V_1 \oplus V_2$ where V_1 is the space of all real matrices of the form

$$\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \tag{3.126}$$

and V_2 is the space of all real matrices of the form

$$\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}. \tag{3.127}$$

Bifurcation Problem

$$\frac{dx}{dt} = f(x,\mu), \quad x \in \mathbf{R}^n, \quad \mu \in \mathbf{R}$$

st. Jacobian $\equiv (df)_{(x_0,\mu)}$ and e-val$\{(df)_{(x_0,\mu)}\} = \{0, \ldots\}$.

Theorem

Γ is a symmetry group of $\dot{x} = f(x,\mu)$ iff $\gamma f(x,\mu) = f(\gamma x,\mu)$.
Equivalently: $M_\gamma f(x,\mu) = f(M_\gamma x,\mu)$

Isotropy Subgroups

The symmetry of a stationary sol x is given by $\Sigma_x = \{\gamma \in \Gamma : \gamma x = x\}$

Fixed Point Subspaces of Subgroup Σ

$$\mathrm{Fix}(\Sigma) = \{x \in V : \sigma x = x, \forall \sigma \in \Sigma\}$$

Theorem 4.5 (Equivariant branching lemma) *Let Γ be a finite group or compact Lie group acting absolutely irreducibly on a real vector space, V, and let*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(\boldsymbol{x}, \mu) \tag{4.52}$$

be a Γ-equivariant bifurcation problem with $f(\mathbf{0}, 0) = \mathbf{0}$ and $Df\big|_{(\mathbf{0},0)} = 0$ that satisfies equation (4.51). If Σ is an isotropy subgroup of Γ, satisfying

$$\dim \mathrm{Fix}(\Sigma) = 1, \tag{4.53}$$

then there exists a unique smooth solution branch to $f(\boldsymbol{x}, \mu) = \mathbf{0}$ such that the isotropy subgroup of each solution is Σ.

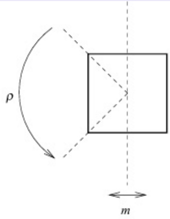


Fig. 4.3. The generators of $\mathcal{D}_4$, the symmetry group of a square, are m, a reflection, and ρ, a rotation through $\pi/2$.

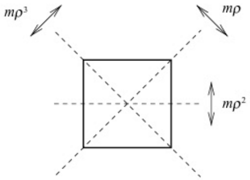


Fig. 4.4. The reflections $m\rho$, $m\rho^2$ and $m\rho^3$ of $\mathcal{D}_4$.

Table 4.1. *The one-dimensional irreps of $\mathcal{D}_4$*

Irrep	e	ρ	ρ^2	ρ^3	m	$m\rho$	$m\rho^2$	$m\rho^3$
R_1	1	1	1	1	1	1	1	1
R_2	1	1	1	1	−1	−1	−1	−1
R_3	1	−1	1	−1	1	−1	1	−1
R_4	1	−1	1	−1	−1	1	−1	1

The natural representation of the group is given by the set of matrices

$$M_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M_\rho = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$M_{\rho^2} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M_{\rho^3} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$M_m = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M_{m\rho} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$M_{m\rho^2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M_{m\rho^3} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

Each irrep leads to a bifurcation problem

$$\frac{dx}{dt} = f(x,\mu), \quad x \in \mathbf{R}^n, \quad \mu \in \mathbf{R}$$

st. $f(0,0) = 0$, $(df)_{(0,0)} = 0$ and $M_\gamma f(x,\mu) = f(M_\gamma x, \mu)$.

R_1

$$1 \cdot f(x,\mu) = f(1 \cdot x, \mu) \Rightarrow \frac{dx}{dt} = \mu + ax^2 + bx\mu + c\mu^2 + \cdots,$$

R_2, R_3, R_4

$$-1 \cdot f(x,\mu) = f(-x,\mu) \Rightarrow \frac{dx}{dt} = \mu x + ax^3 + \cdots,$$

R_2: Z_4 Pattern Formation

$$M_{+1} \cdot v = v, \Rightarrow M_{+1} = \{e, \rho, \rho^2, \rho^3\} = \mathbf{Z}_4,$$
$$M_{-1} \cdot v = v, \Rightarrow M_{-1} = \{m, m\rho, m\rho^2, m\rho^3\}.$$

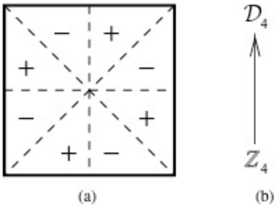


Fig. 4.5. (a) Solution eigenmode for the irrep R_2 of $\mathcal{D}_4$ and (b) isotropy lattice for the irrep R_2 of $\mathcal{D}_4$, where inclusion is shown by an arrow.

R_3 and R_4

$$R_3: \quad \mathbf{Z}_2^2 = \{e, \rho^2, m, m\rho^2\},$$
$$R_4: \quad \mathbf{Z}_2^2 = \{e, \rho^2, m\rho, m\rho^3\}.$$

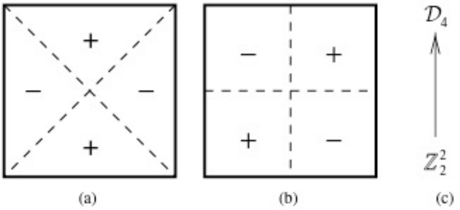


Fig. 4.6. Solution eigenmodes for the irreps (a) R_3 and (b) R_4 of $\mathcal{D}_4$, and (c) the isotropy lattice for the irreps R_3 and R_4 of $\mathcal{D}_4$, where inclusion is shown by an arrow.

for the reflection m shows that

$$\begin{pmatrix} -f_1(x_1, x_2, \mu) \\ f_2(x_1, x_2, \mu) \end{pmatrix} = \begin{pmatrix} f_1(-x_1, x_2, \mu) \\ f_2(-x_1, x_2, \mu) \end{pmatrix} \tag{4.67}$$

must hold, so f_1 must be odd and f_2 must be even in x_1. Applying the rotation matrix

$$M_\rho = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.68}$$

gives

$$\begin{pmatrix} -f_2(x_1, x_2, \mu) \\ f_1(x_1, x_2, \mu) \end{pmatrix} = \begin{pmatrix} f_1(-x_2, x_1, \mu) \\ f_2(-x_2, x_1, \mu) \end{pmatrix}. \tag{4.69}$$

From the second row we have

$$f_1(x_1, x_2, \mu) = f_2(-x_2, x_1, \mu) = f_2(x_2, x_1, \mu) = f_1(x_1, -x_2, \mu), \tag{4.70}$$

since f_2 is even in the first argument. Hence f_1 is even in the second argument. Thus up to cubic order, f_1 must take the form

$$f_1(x_1, x_2, \mu) = \mu x_1 - a_1 x_1^3 - a_2 x_2^2 x_1, \tag{4.71}$$

where a_1 and a_2 are real constants, and μ is the real bifurcation parameter. Now using equation (4.69) we can deduce the normal form to be

$$\begin{pmatrix} \mathrm{d}x_1/\mathrm{d}t \\ \mathrm{d}x_2/\mathrm{d}t \end{pmatrix} = \mu \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - a_1 \begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix} - a_2 \begin{pmatrix} x_2^2 x_1 \\ x_1^2 x_2 \end{pmatrix}. \tag{4.72}$$

If we now make the substitution $\widehat{x_i} = |a_1|^{\frac{1}{2}} x_i$, $\widehat{a_2} = a_2/|a_1|$ and immediately drop the hats, the normal form is transformed to

$$\begin{pmatrix} \mathrm{d}x_1/\mathrm{d}t \\ \mathrm{d}x_2/\mathrm{d}t \end{pmatrix} = \mu \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mp \begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix} - a_2 \begin{pmatrix} x_2^2 x_1 \\ x_1^2 x_2 \end{pmatrix}, \tag{4.73}$$

Pure Modes

$$\begin{aligned}(x_1, 0): \quad \mathbf{Z}_2(m) &= \{e, m\rho^2\},\\ (0, x_2): \quad \mathbf{Z}_2(m) &= \{e, m\}.\end{aligned}$$

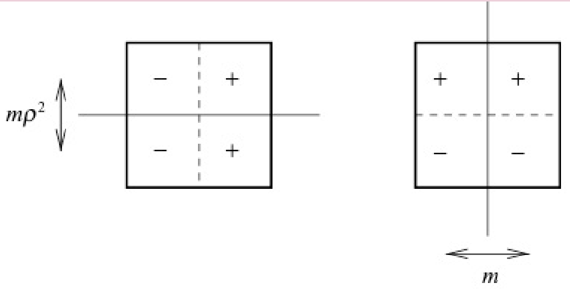


Fig. 4.7. Eigenmodes for the natural representation of $\mathcal{D}_4$.

Mixed Modes

$$
\begin{aligned}
(x_1, x_2 = x_1): \quad \mathbf{Z}_2(m\rho) &= \{e, m\rho\}, \\
(x_1, x_2 = -x_1): \quad \mathbf{Z}_2(m) &= \{e, m\rho^3\}.
\end{aligned}
$$

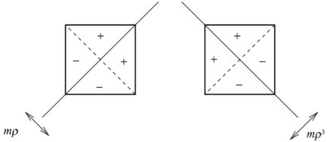


Fig. 4.8. Diagonal modes that have isotropy subgroups with one-dimensional fixed-point subspace in the natural representation of $\mathcal{D}_4$.

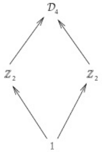


Fig. 4.9. Isotropy lattice for the natural representation of $\mathcal{D}_4$. Inclusion is shown by an arrow.

$$Df = \begin{pmatrix} \mu - 3a_1x_1^2 - a_2x_2^2 & -2a_2x_1x_2 \\ -2a_2x_1x_2 & \mu - 3a_1x_2^2 - a_2x_1^2 \end{pmatrix}. \tag{4.75}$$

Evaluating this at $x_1^2 = \mu/a_1$, $x_2 = 0$, gives the diagonal matrix

$$Df = \begin{pmatrix} -2\mu & 0 \\ 0 & \mu(1 - a_2/a_1) \end{pmatrix}, \tag{4.76}$$

so the solution is stable to perturbations in x_1 if $\mu > 0$ and to perturbations in x_2 if $a_1 < a_2$ (since we must have $\mu/a_1 > 0$ for the solution to exist at all). The matrix is diagonal because we are using coordinates that correspond to the isotypic components with respect to $\Sigma_x = \{e, m\rho^2\}$, the isotropy subgroup of the solution, which are

$$\begin{pmatrix} a \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ b \end{pmatrix}, \quad a, b \in \mathbb{R}. \tag{4.77}$$

The other solution on the group orbit $(0, x_2)$, with $x_2^2 = \mu/a_1$, of course has the same stability properties, but now the nonzero entries in the Jacobian are swapped so that the eigenvalue -2μ corresponds to perturbations in the x_2 direction and $\mu(1 - a_2/a_1)$ to perturbations in the x_1 direction.

If we evaluate the Jacobian at the solution $x_1 = x_2$, with $x_1^2 = \mu/(a_1 + a_2)$, we get

$$Df = \begin{pmatrix} -2a_1\mu/(a_1 + a_2) & -2a_2\mu/(a_1 + a_2) \\ -2a_2\mu/(a_1 + a_2) & -2a_1\mu/(a_1 + a_2) \end{pmatrix}, \tag{4.78}$$

which is not diagonal. This is because the isotypic components with respect to the isotropy subgroup in this case, $\Sigma_x = \{e, m\rho\}$, are different, namely

$$\begin{pmatrix} a \\ a \end{pmatrix} \text{ and } \begin{pmatrix} b \\ -b \end{pmatrix}, \quad a, b \in \mathbb{R}. \tag{4.79}$$

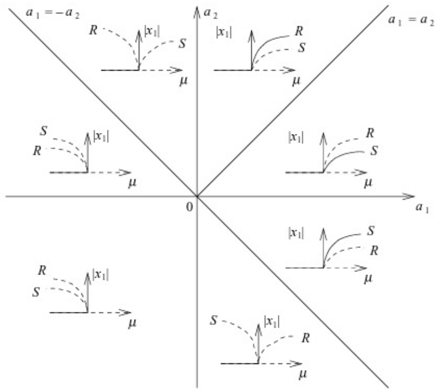


Fig. 4.10. Bifurcation diagrams for the bifurcation in a square box for various values of the coefficients a_1 and a_2 in equation (4.72). Solid lines represent stable solutions and dashed lines unstable solutions. Branches of solutions of type $(x, 0)$ and (x, x) are labelled R and S respectively.

Planar Lattice

A planar lattice $\mathcal{L}$ is generated by two linearly independent vectors l_1 and $l_2 \in \mathbf{R}^2$ is

$$\mathcal{L} = \{\vec{l} = n_1 \vec{l}_1 + n_2 \vec{l}_2\}$$

Dual Lattice $\mathcal{L}^*$

$$\mathcal{L}^* = \{n_1 \vec{k}_1 + n_2 \vec{k}_2 : \vec{k}_i \cdot \vec{l}_j = 2\pi\delta_{ij}, i, j = 1, 2\}$$

Fourier Modes

$$u(\mathbf{x}, t) = \sum_{k \in \mathcal{L}^*} z_k(t) e^{i\vec{k}\cdot\vec{x}} + c.c.$$

Some possible convection planforms

(a) (b) (c)

Figure: a) Stripes or rolls, b) squares and c) hexagons. Constructed from filled contour plots of $u = \sum_j (e^{i\mathbf{k}_j\cdot\mathbf{x}} + e^{-i\mathbf{k}_j\cdot\mathbf{x}})$ for a) $\mathbf{k}_1 = (1,0)$, b) $\mathbf{k}_1 = (1,0)$, $\mathbf{k}_2 = (0,1)$, and c) $\mathbf{k}_1 = (1,0)$, $\mathbf{k}_2 = (-1/2, \sqrt{3}/2)$, $\mathbf{k}_3 = (-1/2, -\sqrt{3}/2)$.

We now allow the pattern to vary in both horizontal directions, x_1 and x_2, but impose periodicity length $L = \lambda_c = 2\pi$ in both of these directions, so that the domain is a periodically repeating square. The eigenvectors are then $e^{\pm ix_1}$ and $e^{\pm ix_2}$. We write nonlinear solutions as

$$u(x_1, x_2, t) = z_1(t)e^{ix_1} + \bar{z}_1(t)e^{-ix_1} + z_2(t)e^{ix_2} + \bar{z}_2(t)e^{-ix_2} \quad (8)$$

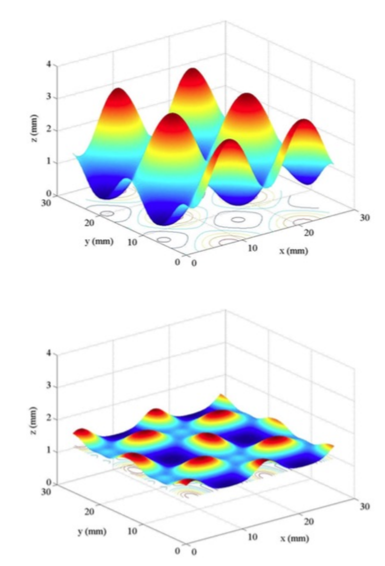
We seek equations that are equivariant with respect to the group generated by rotations by angle $\pi/2$ and reflection in x_1 (i.e. the group D_4), and also by translations by $\boldsymbol{p} = (p_1, p_2)$ in x_1 and x_2 (the two-torus T^2).

$$\begin{aligned}
S_{\pi/2}u(x_1, x_2, t) &\equiv u(x_2, -x_1, t) = z_1(t)e^{ix_2} + \bar{z}_1(t)e^{-ix_2} + z_2(t)e^{-ix_1} + \bar{z}_2(t)e^{ix_1} && \text{(9a)}\\
\kappa u(x_1, x_2, t) &\equiv u(-x_1, x_2, t) = z_1(t)e^{-ix_1} + \bar{z}_1(t)e^{ix_1} + z_2(t)e^{ix_2} + \bar{z}_2(t)e^{-ix_2} && \text{(9b)}\\
P_{p_1,p_2}u(x_1, x_2, t) &\equiv u(x_1 + p_1, x_2 + p_2, t) && \text{(9c)}\\
&= z_1(t)e^{i(x_1+p_1)} + \bar{z}_1(t)e^{-i(x_1+p_1)} + z_2(t)e^{i(x_2+p_2)} + \bar{z}_2(t)e^{-i(x_2+p_2)} && \text{(9d)}
\end{aligned}$$

leading us to define the action of these operators on the amplitudes as:

$$\begin{aligned}
S_{\pi/2}(z_1, z_2) &\equiv (\bar{z}_2, z_1) && \text{(10a)}\\
\kappa(z_1, z_2) &\equiv (\bar{z}_1, z_2) && \text{(10b)}\\
P_{p_1,p_2}(z_1, z_2) &\equiv (e^{ip_1}z_1, e^{ip_2}z_2) && \text{(10c)}
\end{aligned}$$

$$\begin{aligned}
\dot{z}_1 &= \mu z_1 - (a_1|z_1|^2 + a_2|z_2|^2)z_1\\
\dot{z}_2 &= \mu z_2 - (a_2|z_1|^2 + a_1|z_2|^2)z_2
\end{aligned}$$


z (mm)
y (mm)
x (mm)
z (mm)
y (mm)
x (mm)

1.4 Hexagons

We now consider a hexagonal lattice. We define three wavevectors $\boldsymbol{k}_j = (\cos 2\pi(j-1)/3, \sin 2\pi(j-1)/3)$ oriented at angles of 120° to one another, as in figure 7. We write solutions

$$u(x,y,t) = z_1(t)e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}} + z_2(t)e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}} + z_3(t)e^{i\boldsymbol{k}_3\cdot\boldsymbol{x}} + c.c. \tag{14}$$

and seek equations of evolution for (z_1, z_2, z_3) that are equivariant under the group generated by rotation by $2\pi/3$ and reflections (i.e. the group D_6), as well as by translations by $\boldsymbol{p}$ (the group T^2).

Calculations similar to those for the case of the square lattice lead us to define the action of these operators on the amplitudes as:

$$S_{2\pi/3}(z_1, z_2, z_3) \equiv (z_3, z_1, z_2) \tag{15}$$

$$\kappa(z_1, z_2, z_3) \equiv (\bar{z}_1, \bar{z}_2, \bar{z}_3) \tag{16}$$

$$P_{\boldsymbol{p}}(z_1, z_2, z_3) \equiv (e^{i\boldsymbol{k}_1\cdot\boldsymbol{p}}z_1, e^{i\boldsymbol{k}_2\cdot\boldsymbol{p}}z_2, e^{i\boldsymbol{k}_3\cdot\boldsymbol{p}}z_3) \tag{17}$$

Contrary to the square case, the hexagonal case allows quadratic terms, since $\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 = 0$. Thus, translation by $\boldsymbol{p}$ of (z_1, z_2, z_3) transforms the term $\bar{z}_2\bar{z}_3$ as follows:

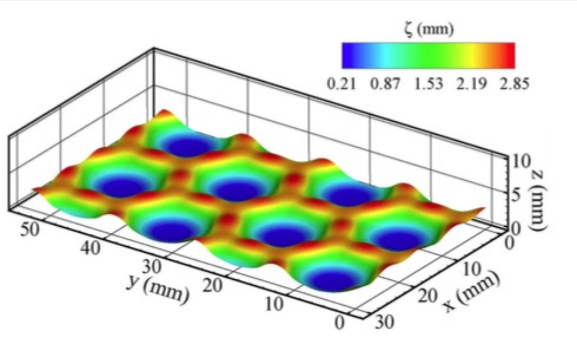
$$\bar{z}_2\bar{z}_3 \to e^{-i\boldsymbol{k}_2\cdot\boldsymbol{p}}\bar{z}_2 e^{-i\boldsymbol{k}_3\cdot\boldsymbol{p}}\bar{z}_3 = e^{-i(\boldsymbol{k}_2+\boldsymbol{k}_3)\cdot\boldsymbol{p}}\bar{z}_2\bar{z}_3 = e^{i\boldsymbol{k}_1\cdot\boldsymbol{p}}\bar{z}_2\bar{z}_3 \tag{18}$$

Since this is the same way in which translation by $\boldsymbol{p}$ transforms z_1, the term $\bar{z}_2\bar{z}_3$ can appear in the evolution equation for z_1.

The resulting equivariant equations to cubic order are:

$$\dot{z}_1 = \left(\mu - b|z_1|^2 - c(|z_2|^2 + |z_3|^2)\right) z_1 + a\bar{z}_2\bar{z}_3 \tag{19}$$

and similarly for z_2, z_3, with real coefficients.


ζ (mm)
0.21 0.87 1.53 2.19 2.85
10
5
0
z (mm)
50
40
30
20
10
0
y (mm)
0
10
20
30
x (mm)

THANK YOU