M636 MATHEMATICAL MODELING

Midterm Exam – Fall 2015

Name: ____

Instructions

1. Exams are considered, strictly, individual: all work should be your own, carried out individually, not in a group. Any form of cheating (including copying someone else work or allowing to be copied) will result in an "F" in the final grade of the course and a referral to the Dean's office for further action.
2. SHOW ALL WORK..
3. Graphical and/or computational work can also be included in support of the analysis. But unless otherwise stated in the problem, answers based purely on graphical and/or computational work are considered incomplete.
4. Include only the most appropriate plots and STAPLE all pages.

I, ____________________________, pledge that this exam is completely my own work, and that I did not take, borrow or steal any portions from any other person. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

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Signature

1. Population Dynamics.

The spruce budworm is a serious pest in eastern Canada, where it attacks the leaves of the balsam fir tree. When an outbreak occurs, the budworms can defoliate and kill most of the fir trees in the forest in about four years. In 1978 Ludwig proposed the following model to explain the outbreak of budworms:

$$\frac{dN}{dT} = RN\left(1 - \frac{N}{K}\right) - P(N) \qquad \text{with} \qquad P(N) = \frac{BN^2}{A^2 + N^2} \tag{1}$$

where $A, B, K, R > 0$. The parameter R represents growth rate and K the carrying capacity. The term $P(N)$ represents the death rate due to predators (cf. birds).

(a) Plot a typical graph for $P(N)$ and give an ecological explanation for its shape as well as the meaning of A and B.

(b) By rescaling time: $t = \tau T$ and population: $n = \eta N$, show that (1) can be rewritten as

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{k}\right) - \frac{n^2}{1 + n^2}. \tag{2}$$

Find τ, η, k and r as a function of the original parameters A, B, K, R.

(c) Show that (0,0) is an equilibrium and determine its stability. Show that the remaining equilibria of Eq. (2) happen at the intersection points of $f(n) = \frac{n}{1+n^2}$ and $g(n) = r\left(1 - \frac{n}{k}\right)$. For a fixed value of k, sketch $f(n)$ and the 5 possibilities for $g(n)$ such that, for increasing r, you have (i) 1, (ii) 2, (ii) 3, (iv) 2, and (v) 1 intersections between f and g.

(d) For each of the 5 cases in (c), label the equilibrium points, sketch the phase line and give a detailed description of what you expect from the dynamics of the population. (Do not attempt to find an explicit form for the equilibrium points.)

(e) As mentioned in exercise 2, a bifurcation occurs when an equilibrium point disappears/appears or changes stability. Explain the possible mechanism for a bifurcation to occur in such a way that the populations jump from a small size stable equilibrium to a large size stable equilibrium, i.e. an outbreak.

2. Hopf Bifurcation.

Consider a modified version of the Van der pol model

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = a \tag{3}$$

where μ and a are constant parameters.

(a) Convert the second order ODE to a first-order system of ODE's.

(b) Analytically, calculate *all* equilibrium points and study their stability.

(c) Find the curves in (μ, a) parameter-space at which the eigenvalues of the linearized Jacobian matrix are purely imaginary, which is equivalent to the condition: $trace(J) = 0$ and $det(J) > 0$. This locus of points is called a *Hopf bifurcation.*

(d) Sketch a diagram in the (μ, a) plane illustrating the change in stability of each equilibrium point when both μ and a change. Use pplane6 or any other equivalent software to sketch phase portraits of different types of behaviors.

3. Bursting Behavior.

In response to glucose, β-cells of the pancreatic islet secrete insulin, which causes the increase use or uptake of glucose in target tissues such as muscle, liver, and adipose tissue. When blood levels of glucose decline, insulin secretion stops, and the tissues begin to use their energy stores instead. Interruption of this control system results in diabetes. It is believed that electrical bursting plays an important role in the release of insulin from the cell. A possible mechanism for generating bursting behavior is through the FitzHugh-Nagumo model

$$\begin{cases} \dfrac{dx}{dt} &= y - x^3 + 3x^2 + I - z \\ \dfrac{dy}{dt} &= 1 - 5x^2 - y \\ \dfrac{dz}{dt} &= r\left[s\left(x + \dfrac{1+\sqrt{5}}{2}\right) - z\right], \end{cases} \tag{4}$$

where I is an input current applied to the cell, and r and s are positive parameters.

(a) Assume $I = 0$, $z = 0$ and $r = 0$. Observe that the first two equations decouple from the last equation. Find *all* equilibrium points of the first two equations.

(b) Determine the local stability properties of *all* equilibrium points in part (a), and sketch the phase plane (xy-plane) solutions using eigenvalues, eigenvectors,, and nullcline curves.

(c) Repeat (a) and (b) with $I = 0.4$, $I = 2$, $I = 4$, and various values of z. Explain how the phase plane solutions change as I and z change.

(d) Write a MATLAB program to integrate the full system of equations with the same values of input current: $I = 0.4$, $I = 2$, and $I = 4$. For each value of I, produce the following graphs: (1) Time series of $x(t)$, $y(t)$, and $z(t)$, all in the same plot; (2) Phase plane diagrams: x vs y, x vs z, and y vs z, all in the same plot.

(e) Write a brief explanation of your results relevant to the biological interpretation of the model.

4. Solve Exercise 34.5 from the textbook.