M636 MATHEMATICAL MODELING

Final Exam – Fall 2015

Name: ______________________________

Instructions

1. Exams are considered, strictly, individual: all work should be your own, carried out individually, not in a group. Any form of cheating (including copying someone else work or allowing to be copied) will result in an "F" in the final grade of the course and a referral to the Dean's office for further action.
2. SHOW ALL WORK..
3. Graphical and/or computational work can also be included in support of the analysis. But unless otherwise stated in the problem, answers based purely on graphical and/or computational work are considered incomplete.
4. Include only the most appropriate plots and STAPLE all pages.

I, ________________________, pledge that this exam is completely my own work, and that I did not take, borrow or steal any portions from any other person. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

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Signature

1. (25 Pts) Swift-Hohenberg Model on a Finite Line

The Swift-Hohenberg equation (1977) was originally proposed as a simplified model of convective instability in a one-dimensional system. It takes the form

$$\frac{\partial u}{\partial t} = \left[\mu - (\nabla^2 + 1)^2\right] u - u^3, \tag{1}$$

where $u = u(x,t)$. Assume the following boundary conditions

$$u = \frac{\partial^2 u}{\partial x^2}, \qquad x = 0,\ L.$$

Study the stability properties of the homogeneous trivial solution $u_0 = 0$ by performing the following tasks:

(i) Linearize (1) about $u = u_0$: Set $u = u_0 + \tilde{u}$ and write the linearized equation for $\tilde{u}$.

(ii) Let the perturbation $\tilde{u}$ be defined as $\tilde{u} = e^{\sigma t + ikx}$. Substitute into the linearized equation for $\tilde{u}$ and find the characteristic equation for the critical or marginal eigenvalues σ as a function of μ and k.

(iii) Solve linearized equation for $\tilde{u}$ using separation of variables: Set $\tilde{u}(x,t) = G(t)\phi(x)$, substitute and divide across by $G(t)\phi(x)$. Set separated variables to a common constant σ and then solve the resulting eigenvalue problems for $G(t)$ and $\phi(x)$.

(iv) Compute marginal stability curves $\sigma_n = 0$ and study the behavior of the perturbation for various values of n in the (μ, L) plane.

2. (25 Pts) Restrictions Imposed by Symmetry

Consider the system

$$\begin{aligned} \frac{dx}{dt} &= f(x,y,z) \\ \frac{dy}{dt} &= g(x,y,z) \\ \frac{dz}{dt} &= h(x,y,z). \end{aligned} \tag{2}$$

under the symmetries:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y \\ z \\ x \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ z \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix},$$

Show that these symmetries imply

$$\begin{aligned} h(y,z,x) &= g(z,x,y) = f(x,y,z) \\ f(x,y,z) &= f(x,z,y) \\ f(x,y,z) &= -f(-x,-y,-z) \end{aligned}$$

and hence write down the series expansion for $f(x,y,z)$, $g(x,y,z)$ and $h(x,y,z)$ for small $|x|$, $|y|$ and $|z|$ to quadratic order.

3. (25 Pts) Steady-State Bifurcation in a Triangle

(a) Find all the absolutely irreducible representations of D_3, the symmetry group of an equilateral triangle.

(b) Work out all the possible solutions that are guaranteed at a steady bifurcation with D_3 symmetry under one other of these representations.

(c) Draw examples of the eigenmodes in an appropriate triangular box, and derive the relevant normal form equations.

(d) Study the stability properties of each bifurcating pattern.

NOTE: All calculations must be carried out by hand. You are not allowed to use any type of mathematical software to solve this problem.

4. (25 Pts) Hopf Bifurcation on a One-Dimensional Lattice

Consider the amplitude equations that govern the time-evolution of one-dimensional waves over a one-dimensional lattice:

$$\begin{aligned}\frac{dz_1}{dt} &= \mu z_1 - (\alpha + \beta i)|z_1|^2 z_1 - (\gamma + \delta i)|z_2|^2 z_1 \\ \frac{dz_2}{dt} &= \mu z_2 - (\alpha + \beta i)|z_2|^2 z_2 - (\gamma + \delta i)|z_1|^2 z_2.\end{aligned} \tag{3}$$

Perform the following tasks:

(i) Write a MATLAB (or equivalent software) code to integrate the amplitude equations over a 1D grid of the form $x \in [a, b]$. For instance, $a = -50$, $b = 50$. Turn in a copy of the code. Note: It's easier and more convenient to write the code using the complex-valued variables in (3), i.e., $(z_1,\ z_2) \in \mathbf{C}^2$.

(ii) Illustrate the stability properties of traveling wave solutions (both left-traveling and right traveling waves) and of standing waves. To do this, assign appropriate values to parameters: $(\mu, \alpha, \beta, \gamma, \delta)$ and initial conditions. For each case, (left-TW, right-TW, and SW) turn in ONE plot of the resulting pattern $u(x, t)$, where:

$$u(x, t) = z_1(t)e^{(x-t)i} + z_2(t)e^{-(x+t)i} + c.c.$$