The subspaces were so named because orbits starting in E^s decayed to zero as t (resp. n for maps) $\uparrow \infty$, orbits starting in E^u became unbounded as t (resp. n for maps) $\uparrow \infty$, and orbits starting in E^c neither grew nor decayed exponentially as t (resp. n for maps) $\uparrow \infty$.

If we suppose that $E^u = \emptyset$, then we find that any orbit will rapidly decay to E^c. Thus, if we are interested in long-time behavior (i.e., stability) we need only to investigate the system restricted to E^c.

It would be nice if a similar type of "reduction principle" applied to the study of the stability of nonhyperbolic fixed points of nonlinear vector fields and maps, namely, that there were an invariant *center manifold* passing through the fixed point to which the system could be restricted in order to study its asymptotic behavior in the neighborhood of the fixed point. That this is the case is the content of the center manifold theory.

2.1A Center Manifolds for Vector Fields

We will begin by considering center manifolds for vector fields. The set-up is as follows. We consider vector fields of the following form

$$\begin{aligned} \dot{x} &= Ax + f(x,y), \\ \dot{y} &= By + g(x,y), \qquad (x,y) \in \mathbb{R}^c \times \mathbb{R}^s, \end{aligned} \tag{2.1.2}$$

where

$$\begin{aligned} f(0,0) &= 0, \qquad Df(0,0) = 0, \\ g(0,0) &= 0, \qquad Dg(0,0) = 0. \end{aligned} \tag{2.1.3}$$

(See Section 1.1C for a discussion of how a general vector field is transformed to the form of (2.1.2) in the neighborhood of a fixed point.)

In the above, A is a $c \times c$ matrix having eigenvalues with zero real parts, B is an $s \times s$ matrix having eigenvalues with negative real parts, and f and g are $\mathbf{C}^r$ functions ($r \geq 2$).

Definition 2.1.1 An invariant manifold will be called a center manifold for (2.1.2) if it can locally be represented as follows

$$W^c(0) = \{ (x,y) \in \mathbb{R}^c \times \mathbb{R}^s \mid y = h(x), |x| < \delta, h(0) = 0, Dh(0) = 0 \}$$

for δ sufficiently small.

We remark that the conditions $h(0) = 0$ and $Dh(0) = 0$ imply that $W^c(0)$ is tangent to E^c at $(x,y) = (0,0)$. The following three theorems are taken from the excellent book by Carr [1981].

The first result on center manifolds is an existence theorem.

origin is stable but not asymptotically stable). We will answer the question of stability using center manifold theory.

From Theorem 2.1.1, there exists a center manifold for (2.1.11) which can locally be represented as follows

$$W^c(0) = \{(x, y) \in \mathbb{R}^2 \mid y = h(x), \mid x \mid < \delta, h(0) = Dh(0) = 0\} \tag{2.1.12}$$

for δ sufficiently small. We now want to compute $W^c(0)$. We assume that $h(x)$ has the form

$$h(x) = ax^2 + bx^3 + \mathcal{O}(x^4), \tag{2.1.13}$$

and we substitute (2.1.13) into equation (2.1.10), which $h(x)$ must satisfy to be a center manifold. We then equate equal powers of x, and in that way we can compute $h(x)$ to any desired order of accuracy. In practice, computing only a few terms is usually sufficient to answer questions of stability.

We recall from (2.1.10) that the equation for the center manifold is given by

$$\mathcal{N}(h(x)) = Dh(x)\left[Ax + f(x, h(x))\right] - Bh(x) - g(x, h(x)) = 0, \tag{2.1.14}$$

where, in this example, we have $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned} A &= 0, \\ B &= -1, \\ f(x, y) &= x^2 y - x^5, \\ g(x, y) &= x^2. \end{aligned} \tag{2.1.15}$$

Substituting (2.1.13) into (2.1.14) and using (2.1.15) gives

$$\begin{aligned} \mathcal{N}(h(x)) = (2ax + 3bx^2 + \cdots)&(ax^4 + bx^5 - x^5 + \cdots) \\ &+ ax^2 + bx^3 - x^2 + \cdots = 0. \end{aligned} \tag{2.1.16}$$

In order for (2.1.16) to hold, the coefficients of each power of x must be zero; see Exercise 2.20. Thus, equating coefficients on each power of x to zero gives

$$\begin{aligned} &x^2 : a - 1 = 0 \Rightarrow a = 1, \\ &x^3 : b = 0, \\ &\ \vdots \quad \vdots \end{aligned} \tag{2.1.17}$$

and we therefore have

$$h(x) = x^2 + \mathcal{O}(x^4). \tag{2.1.18}$$

Using (2.1.18) along with Theorem 2.1.1, the vector field restricted to the center manifold is given by

$$\dot{x} = x^4 + \mathcal{O}(x^5). \tag{2.1.19}$$

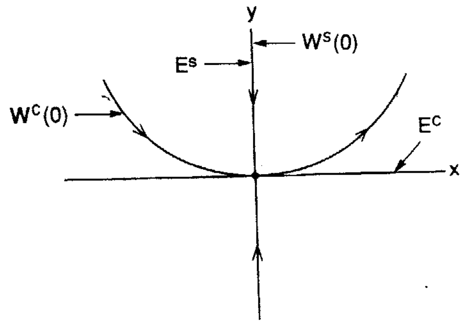

FIGURE 2.1.1.

For x sufficiently small, $x = 0$ is thus unstable in (2.1.19). Hence, by Theorem 2.1.1, $(x, y) = (0, 0)$ is unstable in (2.1.11); see Figure 2.1.1 for an illustration of the geometry of the flow near $(x, y) = (0, 0)$.

This example illustrates an important phenomenon, which we now describe.

The Failure of the Tangent Space Approximation

The idea is as follows. Consider (2.1.11). One might expect that the y components of orbits starting near $(x, y) = (0, 0)$ should decay to zero exponentially fast. Therefore, the question of stability of the origin should reduce to a study of the x component of orbits starting near the origin. One might thus be very tempted to set $y = 0$ in (2.1.11) and study the reduced equation

$$\dot{x} = -x^5. \tag{2.1.20}$$

This corresponds to approximating $W^c(0)$ by E^c. However, $x = 0$ is stable for (2.1.20) and, therefore, we would arrive at the *wrong* conclusion that $(x, y) = (0, 0)$ is stable for (2.1.20). The tangent space approximation might sometimes work, but, as this example shows, it does not always do so.

2.1B Center Manifolds Depending on Parameters

Suppose (2.1.2) depends on a vector of parameters, say $\varepsilon \in \mathbb{R}^p$. In this case we write (2.1.2) in the form

$$\begin{aligned} \dot{x} &= Ax + f(x, y, \varepsilon), \\ \dot{y} &= By + g(x, y, \varepsilon), \end{aligned} \qquad (x, y, \varepsilon) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^p, \quad (2.1.21)$$

where

$$\begin{aligned} f(0,0,0) &= 0, \qquad Df(0,0,0) = 0, \\ g(0,0,0) &= 0, \qquad Dg(0,0,0) = 0, \end{aligned}$$

and we have the same assumptions on A and B as in (2.1.2), with f and g also being $\mathbf{C}^r$ ($r \geq 2$) functions in some neighborhood of $(x, y, \varepsilon) = (0, 0, 0)$. An obvious question is why do we not allow the matrices A and B to depend on ε? This will be answered shortly.

The way in which we will handle parametrized systems is to include the parameter ε as a new *dependent variable* as follows

$$\begin{aligned} \dot{x} &= Ax + f(x, y, \varepsilon), \\ \dot{\varepsilon} &= 0, \\ \dot{y} &= By + g(x, y, \varepsilon), \end{aligned} \qquad (x, \varepsilon, y) \in \mathbb{R}^c \times \mathbb{R}^p \times \mathbb{R}^s. \tag{2.1.22}$$

At first glance it might appear that nothing is really gained from this action, but we will argue otherwise.

Let us suppose we are considering (2.1.22) afresh. It obviously has a fixed point at $(x, \varepsilon, y) = (0, 0, 0)$. The matrix associated with the linearization of (2.1.22) about this fixed point has $c + p$ eigenvalues with zero real part and s eigenvalues with negative real part. Now let us apply center manifold theory. Modifying Definition 2.1.1, a center manifold will be represented as a graph over the *x and ε variables*, i.e., the graph of $h(x, \varepsilon)$ for x and ε sufficiently small. Theorem 2.1.1 still applies, with the vector field reduced to the center manifold given by

$$\begin{aligned} \dot{u} &= Au + f(u, h(u, \varepsilon), \varepsilon), \\ \dot{\varepsilon} &= 0, \end{aligned} \qquad (u, \varepsilon) \in \mathbb{R}^c \times \mathbb{R}^p. \tag{2.1.23}$$

Theorems 2.1.2 and 2.1.3 also follow (we will worry about any modifications to computing the center manifold shortly). Thus, adding the parameter as a new dependent variable merely acts to augment the matrix A in (2.1.2) by adding p new center directions that have no dynamics, and the theory goes through just the same. However, there is a new concept which will be important when we study *bifurcation theory*; namely, the center manifold exists for all ε in a sufficiently small neighborhood of $\varepsilon = 0$. We will learn in Chapter 3 that it is possible for solutions to be created or destroyed by perturbing nonhyperbolic fixed points. Thus, since the invariant center manifold exists in a sufficiently small neighborhood in both x and ε of $(x, \varepsilon) = (0, 0)$, all bifurcating solutions will be contained in the lower dimensional center manifold.

Let us now worry about computing the center manifold. From the existence theorem for center manifolds, locally we have

$$\begin{aligned} W^c_{\text{loc}}(0) = \{ (x, \varepsilon, y) \in \mathbb{R}^c \times \mathbb{R}^p \times \mathbb{R}^s \mid y = h(x, \varepsilon), |x| < \delta, \\ |\varepsilon| < \bar{\delta},\ h(0, 0) = 0,\ Dh(0, 0) = 0 \} \end{aligned} \tag{2.1.24}$$

for δ and $\bar{\delta}$ sufficiently small. Using invariance of the graph of $h(x, \varepsilon)$ under the dynamics generated by (2.1.22) we have

$$\dot{y} = D_x h(x, \varepsilon)\dot{x} + D_\varepsilon h(x, \varepsilon)\dot{\varepsilon} = Bh(x, \varepsilon) + g\left(x, h(x, \varepsilon), \varepsilon\right). \tag{2.1.25}$$

However,

$$\begin{aligned} \dot{x} &= Ax + f\left(x, h(x,\varepsilon), \varepsilon\right), \\ \dot{\varepsilon} &= 0; \end{aligned} \tag{2.1.26}$$

hence substituting (2.1.26) into (2.1.25) results in the following quasilinear partial differential equation that $h(x,\varepsilon)$ must satisfy in order for its graph to be a center manifold.

$$\begin{aligned} \mathcal{N}\left(h(x,\varepsilon)\right) = D_x h(x,\varepsilon)\left[Ax + f\left(x, h(x,\varepsilon), \varepsilon\right)\right] \\ - Bh(x,\varepsilon) - g\left(x, h(x,\varepsilon), \varepsilon\right) = 0. \end{aligned} \tag{2.1.27}$$

Thus, we see that (2.1.27) is very similar to (2.1.10).

Before considering a specific example we want to point out an important fact. By considering ε as a new dependent variable, terms such as

$$x_i \varepsilon_j, \qquad 1 \le i \le c, \quad 1 \le j \le p,$$

or

$$y_i \varepsilon_j, \qquad 1 \le i \le s, \quad 1 \le j \le p,$$

become *nonlinear terms*. In this case, returning to a question asked at the beginning of this section, the parts of the matrices A and B depending on ε are now viewed as nonlinear terms and are included in the f and g terms of (2.1.22), respectively. We remark that in applying center manifold theory to a given system, it must first be transformed into the standard form (either (2.1.2) or (2.1.22)).

EXAMPLE 2.1.2 Consider the Lorenz equations

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= \bar{\rho}x + x - y - xz, \qquad (x,y,z) \in \mathbb{R}^3, \\ \dot{z} &= -\beta z + xy, \end{aligned} \tag{2.1.28}$$

where σ and β are viewed as fixed positive constants and $\bar{\rho}$ is a parameter (note: in the standard version of the Lorenz equations it is traditional to put $\bar{\rho} = \rho - 1$). It should be clear that $(x,y,z) = (0,0,0)$ is a fixed point of (2.1.28). Linearizing (2.1.28) about this fixed point, we obtain the associated matrix

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}. \tag{2.1.29}$$

(Note: recall, $\bar{\rho}x$ is a nonlinear term.)

Since (2.1.29) is in block form, the eigenvalues are particularly easy to compute and are given by

$$0, -\sigma - 1, -\beta, \tag{2.1.30}$$

with eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.1.31}$$

Our goal is to determine the nature of the stability of $(x, y, z) = (0, 0, 0)$ for $\bar{\rho}$ near zero. First, we must put (2.1.29) into the standard form (2.1.22). Using the eigenbasis (2.1.31), we obtain the transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{2.1.32}$$

with inverse

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{1+\sigma} \begin{pmatrix} 1 & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1+\sigma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{2.1.33}$$

which transforms (2.1.28) into

$$\begin{aligned} \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(1+\sigma) & 0 \\ 0 & 0 & -\beta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ &\quad + \frac{1}{1+\sigma} \begin{pmatrix} \sigma\bar{\rho}(u+\sigma v) - \sigma w(u+\sigma v) \\ -\bar{\rho}(u+\sigma v) + w(u+\sigma v) \\ (1+\sigma)(u+\sigma v)(u-v) \end{pmatrix}, \\ \dot{\bar{\rho}} &= 0. \end{aligned} \tag{2.1.34}$$

Thus, from center manifold theory, the stability of $(x, y, z) = (0, 0, 0)$ near $\bar{\rho} = 0$ can be determined by studying a one-parameter family of first-order ordinary differential equations on a center manifold, which can be represented as a graph over the u and $\bar{\rho}$ variables, i.e.,

$$\begin{aligned} W^c(0) = \{ (u, v, w, \bar{\rho}) \in \mathbb{R}^4 \mid v = h_1(u, \bar{\rho}), w = h_2(u, \bar{\rho}), \\ h_i(0,0) = 0, Dh_i(0,0) = 0, i = 1, 2 \} \end{aligned} \tag{2.1.35}$$

for u and $\bar{\rho}$ sufficiently small.

We now want to compute the center manifold and derive the vector field on the center manifold. Using Theorem 2.1.3, we assume

$$\begin{aligned} h_1(u, \bar{\rho}) &= a_1 u^2 + a_2 u\bar{\rho} + a_3 \bar{\rho}^2 + \cdots, \\ h_2(u, \bar{\rho}) &= b_1 u^2 + b_2 u\bar{\rho} + b_3 \bar{\rho}^2 + \cdots. \end{aligned} \tag{2.1.36}$$

Recall from (2.1.27) that the center manifold must satisfy

$$\begin{aligned} \mathcal{N}\big(h(x, \varepsilon)\big) = D_x h(x, \varepsilon) \left[Ax + f(x, h(x, \varepsilon), \varepsilon)\right] \\ - Bh(x, \varepsilon) - g(x, h(x, \varepsilon), \varepsilon) = 0, \end{aligned} \tag{2.1.37}$$

where, in this example,

$$
\begin{aligned}
&x \equiv u, \qquad y \equiv (v, w), \qquad \varepsilon \equiv \bar{\rho}, \qquad h = (h_1, h_2),\\
&A = 0,\\
&B = \begin{pmatrix} -(1+\sigma) & 0 \\ 0 & -\beta \end{pmatrix},\\
f(x, y, \varepsilon) &= \frac{1}{1+\sigma}[\sigma\bar{\rho}(u+\sigma v) - \sigma w(u+\sigma v)],\\
g(x, y, \varepsilon) &= \frac{1}{1+\sigma}\begin{pmatrix} -\bar{\rho}(u+\sigma v) + w(u+\sigma v) \\ (1+\sigma)(u+\sigma v)(u-v) \end{pmatrix}.
\end{aligned}
\tag{2.1.38}
$$

Substituting (2.1.36) into (2.1.37) and using (2.1.38) gives the two components of the equation for the center manifold.

$$
\begin{aligned}
&(2a_1 u + a_2\bar{\rho} + \cdots)\left[\frac{\sigma}{1+\sigma}\left(\bar{\rho}(u+\sigma h_1) - h_2(u+\sigma h_1)\right)\right]\\
&\quad + (1+\sigma)h_1 + \frac{\bar{\rho}}{1+\sigma}(u+\sigma h_1) - \frac{h_2}{1+\sigma}(u+\sigma h_1) = 0,\\
&(2b_1 u + b_2\bar{\rho} + \cdots)\left[\frac{\sigma}{1+\sigma}\left(\bar{\rho}(u+\sigma h_1) - h_2(u+\sigma h_1)\right)\right]\\
&\quad + \beta h_2 - (u+\sigma h_1)(u-h_1) = 0.
\end{aligned}
\tag{2.1.39}
$$

Equating terms of like powers to zero gives

$$
\begin{aligned}
u^2 &: a_1(1+\sigma) = 0 \Rightarrow a_1 = 0,\\
&\beta b_1 - 1 = 0 \Rightarrow b_1 = \frac{1}{\beta},\\
u\bar{\rho} &: (1+\sigma)a_2 + \frac{1}{1+\sigma} = 0 \Rightarrow a_2 = \frac{-1}{(1+\sigma)^2},\\
&\beta b_2 = 0 \Rightarrow b_2 = 0.
\end{aligned}
\tag{2.1.40}
$$

Then, using (2.1.40) and (2.1.36), we obtain

$$
\begin{aligned}
h_1(u, \bar{\rho}) &= -\frac{1}{(1+\sigma)^2}u\bar{\rho} + \cdots,\\
h_2(u, \bar{\rho}) &= \frac{1}{\beta}u^2 + \cdots.
\end{aligned}
\tag{2.1.41}
$$

Finally, substituting (2.1.41) into (2.1.34) we obtain the vector field reduced to the center manifold

$$
\begin{aligned}
\dot{u} &= \frac{u}{1+\sigma}\left(\sigma\bar{\rho} - \frac{\sigma}{\beta}u^2 + \cdots\right),\\
\dot{\bar{\rho}} &= 0.
\end{aligned}
\tag{2.1.42}
$$