develop the theory for maps, the results will have immediate applications to periodic orbits of vector fields by considering the associated Poincaré map (cf. Section 1.2A).

2. In general, the coordinate transformations will be nonlinear functions of the dependent variables. However, the important point is that these coordinate transformations are found by solving a sequence of *linear* problems.

3. The structure of the normal form is determined entirely by the nature of the linear part of the vector field.

We now begin the development of the method.

2.2A Normal Forms for Vector Fields

Consider the vector field

$$\dot{w} = G(w), \qquad w \in \mathbb{R}^n, \tag{2.2.1}$$

where G is $\mathbf{C}^r$, with r to be specified as we go along (note: in practice we will need $r \geq 4$). Suppose (2.2.1) has a fixed point at $w = w_0$. We first want to perform a few simple (linear) coordinate transformations that will put (2.2.1) into a form which is easier to work with.

1. First we transform the fixed point to the origin by the translation

$$v = w - w_0, \qquad v \in \mathbb{R}^n,$$

under which (2.2.1) becomes

$$\dot{v} = G(v + w_0) \equiv H(v). \tag{2.2.2}$$

2. We next "split off" the linear part of the vector field and write (2.2.2) as follows

$$\dot{v} = DH(0)v + \bar{H}(v), \tag{2.2.3}$$

where $\bar{H}(v) \equiv H(v) - DH(0)v$. It should be clear that $\bar{H}(v) = \mathcal{O}(|v|^2)$.

3. Finally, let T be the matrix that transforms the matrix $DH(0)$ into (real) Jordan canonical form. Then, under the transformation

$$v = Tx, \tag{2.2.4}$$

(2.2.3) becomes

$$\dot{x} = T^{-1}DH(0)Tx + T^{-1}\bar{H}(Tx). \tag{2.2.5}$$

Denoting the (real) Jordan canonical form of $DH(0)$ by J, we have

$$J \equiv T^{-1} DH(0) T, \tag{2.2.6}$$

and we define

$$F(x) \equiv T^{-1} \bar{H}(Tx)$$

so that (2.2.4) is alternately written as

$$\dot{x} = Jx + F(x), \qquad x \in \mathbb{R}^n. \tag{2.2.7}$$

We remark that the transformation (2.2.4) has simplified the linear part of (2.2.3) as much as possible. We now begin the task of simplifying the nonlinear part, $F(x)$.

First, we Taylor expand $F(x)$ so that (2.2.7) becomes

$$\dot{x} = Jx + F_2(x) + F_3(x) + \cdots + F_{r-1}(x) + \mathcal{O}(|x|^r), \tag{2.2.8}$$

where $F_i(x)$ represent the order i terms in the Taylor expansion of $F(x)$. We next introduce the coordinate transformation

$$x = y + h_2(y), \tag{2.2.9}$$

where $h_2(y)$ is second order in y. Substituting (2.2.9) into (2.2.8) gives

$$\begin{aligned} \dot{x} = \big(\mathrm{id} + Dh_2(y)\big)\dot{y} &= Jy + Jh_2(y) + F_2\big(y + h_2(y)\big) \\ &\quad + F_3\big(y + h_2(y)\big) + \cdots + F_{r-1}\big(y + h_2(y)\big) + \mathcal{O}(|y|^r), \end{aligned} \tag{2.2.10}$$

where "id" denotes the $n \times n$ identity matrix. Note that each term

$$F_k\big(y + h_2(y)\big), \qquad 2 \le k \le r-1, \tag{2.2.11}$$

can be written as

$$F_k(y) + \mathcal{O}(|y|^{k+1}) + \cdots + \mathcal{O}(|y|^{2k}), \tag{2.2.12}$$

so that (2.2.10) becomes

$$\begin{aligned} \big(\mathrm{id} + Dh_2(y)\big)\dot{y} = Jy + Jh_2(y) + F_2(y) + \tilde{F}_3(y) \\ + \cdots + \tilde{F}_{r-1}(y) + \mathcal{O}(|y|^r), \end{aligned} \tag{2.2.13}$$

where the terms $\tilde{F}_k(y)$ represent the $\mathcal{O}(|y|^k)$ terms which have been modified due to the coordinate transformation.

Now, for y sufficiently small,

$$\big(\mathrm{id} + Dh_2(y)\big)^{-1} \tag{2.2.14}$$

exists and can be represented in a series expansion as follows (see Exercise 2.7)

$$\left(\mathrm{id} + Dh_2(y)\right)^{-1} = \mathrm{id} - Dh_2(y) + \mathcal{O}(|y|^2). \tag{2.2.15}$$

Substituting (2.2.15) into (2.2.13) gives

$$\begin{aligned} \dot{y} = Jy + Jh_2(y) - Dh_2(y)Jy + F_2(y) + \tilde{F}_3(y) \\ + \cdots + \tilde{F}_{r-1}(y) + \mathcal{O}(|y|^r). \end{aligned} \tag{2.2.16}$$

Up to this point $h_2(y)$ has been completely arbitrary. However, now we will choose a specific form for $h_2(y)$ so as to simplify the $\mathcal{O}(|y|^2)$ terms as much as possible. Ideally, this would mean choosing $h_2(y)$ such that

$$Dh_2(y)Jy - Jh_2(y) = F_2(y), \tag{2.2.17}$$

which would eliminate $F_2(y)$ from (2.2.16). Equation (2.2.17) can be viewed as an equation for the unknown $h_2(y)$. We want to motivate the fact that, when viewed in the correct way, it is in fact a linear equation acting on a linear vector space. This will be accomplished by 1) defining the appropriate linear vector space; 2) defining the linear operator on the vector space; and 3) describing the equation to be solved in this linear vector space (which will turn out to be (2.2.17)). We begin with Step 1.

Step 1. The Space of Vector-Valued Monomials of Degree k, H_k. Let $\{s_1, \cdots, s_n\}$ denote a basis of $\mathbb{R}^n$, and let $y = (y_1, \cdots, y_n)$ be coordinates with respect to this basis. Now consider those basis elements with coefficients consisting of monomials of degree k, i.e.,

$$(y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n}) s_i, \quad \sum_{j=1}^{n} m_j = k, \tag{2.2.18}$$

where $m_j \geq 0$ are integers. We refer to these objects as *vector-valued monomials of degree k*. The set of all vector-valued monomials of degree k forms a linear vector space, which we denote by H_k. An obvious basis for H_k consists of elements formed by considering all possible monomials of degree k that multiply each s_i. The reader should verify these statements. Let us consider a specific example.

EXAMPLE 2.2.1 We consider the standard basis

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.2.19}$$

on $\mathbb{R}^2$ and denote the coordinates with respect to this basis by x and y, respectively. Then we have

$$H_2 = \mathrm{span}\left\{ \begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \end{pmatrix} \right\}. \tag{2.2.20}$$

Step 2. The Linear Map on H_k. Now let us reconsider equation (2.2.17). It should be clear that $h_2(y)$ can be viewed as an element of H_2. The reader should easily be able to verify that the map

$$h_2(y) \longmapsto Dh_2(y)Jy - Jh_2(y) \tag{2.2.21}$$

is a linear map of H_2 into H_2. Indeed, for any element $h_k(y) \in H_k$, it similarly follows that

$$h_k(y) \longmapsto Dh_k(y)Jy - Jh_k(y) \tag{2.2.22}$$

is a linear map of H_k into H_k.

Let us mention some terminology associated with Equation (2.2.17) that has become traditional. Due to its presence in Lie algebra theory (see, e.g., Olver [1986]) this map is often denoted as

$$L_J\big(h_k(y)\big) \equiv -\big(Dh_k(y)Jy - Jh_k(y)\big) \tag{2.2.23}$$

or

$$-\big(Dh_k(y)Jy - Jh_k(y)\big) \equiv [h_k(y), Jy], \tag{2.2.24}$$

where $[\cdot,\cdot]$ denotes the Lie bracket operation on the vector fields $h_k(y)$ and Jy.

Step 3. The Solution of (2.2.17). We now return to the problem of solving (2.2.17). It should be clear that $F_2(y)$ can be viewed as an element of H_2. From elementary linear algebra, we know that H_2 can be (nonuniquely) represented as follows

$$H_2 = L_J(H_2) \oplus G_2. \tag{2.2.25}$$

where G_2 represents a space complementary to $L_J(H_2)$. Solving (2.2.17) is like solving the equation $Ax = b$ from linear algebra. If $F_2(y)$ is in the range of $L_J(\cdot)$, then all $\mathcal{O}(|y|^2)$ terms can be eliminated from (2.2.17). In any case, we can choose $h_2(y)$ so that only $\mathcal{O}(|y|^2)$ terms that are in G_2 remain. We denote these terms by

$$F_2^r(y) \in G_2 \tag{2.2.26}$$

(note: the superscript r in (2.2.26) denotes the term "resonance," which will be explained shortly).

Thus, (2.2.16) can be simplified to

$$\dot{y} = Jy + F_2^r(y) + \tilde{F}_3(y) + \cdots + \tilde{F}_{r-1}(y) + \mathcal{O}(|y|^r). \tag{2.2.27}$$

At this point the meaning of the phrase "simplify the second-order terms" should be clear. It means the introduction of a coordinate change such that, in the new coordinate system, the only second-order terms are in a space complementary to $L_J(H_2)$. If $L_J(H_2) = H_2$, then all second-order terms can be eliminated.

Next let us simplify the $\mathcal{O}(|y|^3)$ terms. Introducing the coordinate change

$$y \longmapsto y + h_3(y), \tag{2.2.28}$$

where $h_3(y) = \mathcal{O}(|y|^3)$ (note: we will retain the same variables y in our equation), and performing the same algebraic manipulations as in dealing with the second-order terms, (2.2.27) becomes

$$\begin{aligned} \dot{y} = Jy + F_2^r(y) + Jh_3(y) - Dh_3(y)Jy + \tilde{F}_3(y) + \tilde{\tilde{F}}_4(y) \\ + \cdots + \tilde{\tilde{F}}_{r-1}(y) + \mathcal{O}(|y|^r), \end{aligned} \tag{2.2.29}$$

where the terms $\tilde{\tilde{F}}_k(y)$, $4 \leq k \leq r-1$, indicate, as before, that the coordinate transformation has modified the terms of order higher than three. Now, simplifying the third-order terms involves solving

$$Dh_3(y)Jy - Jh_3(y) = \tilde{F}_3(y). \tag{2.2.30}$$

The same comments as for second-order terms apply here. The map

$$h_3(y) \longmapsto Dh_3(y)Jy - Jh_3(y) \equiv -L_J(h_3(y)) \tag{2.2.31}$$

is a linear map of H_3 into H_3. Thus, we can write

$$H_3 = L_J(H_3) \oplus G_3, \tag{2.2.32}$$

where G_3 is some space complementary to $L_J(H_3)$. Thus, the third-order terms can be simplified to

$$F_3^r(y) \in G_3. \tag{2.2.33}$$

If $L_J(H_3) = H_3$, then the third-order terms can be eliminated.

Clearly, this procedure can be iterated so that we obtain the following *normal form theorem.*

Theorem 2.2.1 (Normal Form Theorem) *By a sequence of analytic coordinate changes (2.2.8) can be transformed into*

$$\dot{y} = Jy + F_2^r(y) + \cdots + F_{r-1}^r(y) + \mathcal{O}(|y|^r), \tag{2.2.34}$$

where $F_k^r(y) \in G_k$, $2 \leq k \leq r-1$, and G_k is a space complementary to $L_J(H_k)$. Equation (2.2.34) is said to be in normal form.

Several comments are now in order.

1. The terms $F_k^r(y)$, $2 \leq k \leq r-1$, are referred to as *resonance terms* (hence the superscript r). We will explain what this means in Section 2.2D,i).

2. The structure of the nonlinear terms in (2.2.34) is determined entirely by the linear part of the vector field (i.e., J).

3. It should be clear that simplifying the terms at order k does not modify any lower order terms. However, terms of order higher than k are modified. This happens at each step of the application of the method. If one wanted to actually calculate the coefficients on each term of the normal form in terms of the original vector field, it would be necessary to keep track of how the higher order terms are modified by the successive coordinate transformations.

EXAMPLE 2.2.2 We want to compute the normal form for a vector field on $\mathbb{R}^2$ in the neighborhood of a fixed point where the linear part is given by

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.2.35}$$

Second-Order Terms

We have

$$H_2 = \text{span}\left\{ \begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \end{pmatrix} \right\}. \tag{2.2.36}$$

We want to compute $L_J(H_2)$. We do this by computing the action of $L_J(\cdot)$ on each basis element on H_2

$$\begin{aligned}
L_J\begin{pmatrix} x^2 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x^2 \\ 0 \end{pmatrix} - \begin{pmatrix} 2x & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} -2xy \\ 0 \end{pmatrix} = -2\begin{pmatrix} xy \\ 0 \end{pmatrix}, \\
L_J\begin{pmatrix} xy \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} xy \\ 0 \end{pmatrix} - \begin{pmatrix} y & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} -y^2 \\ 0 \end{pmatrix} = -1\begin{pmatrix} y^2 \\ 0 \end{pmatrix}, \\
L_J\begin{pmatrix} y^2 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y^2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 2y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
L_J\begin{pmatrix} 0 \\ x^2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ x^2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2x & 0 \end{pmatrix}\begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} x^2 \\ -2xy \end{pmatrix} \\
&= \begin{pmatrix} x^2 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ xy \end{pmatrix}, \\
L_J\begin{pmatrix} 0 \\ xy \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ xy \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ y & x \end{pmatrix}\begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} xy \\ -y^2 \end{pmatrix} \\
&= \begin{pmatrix} xy \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ y^2 \end{pmatrix}, \\
L_J\begin{pmatrix} 0 \\ y^2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ y^2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 2y \end{pmatrix}\begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} y^2 \\ 0 \end{pmatrix}.
\end{aligned} \tag{2.2.37}$$

From (2.2.37) we have

$$L_J(H_2) = \text{span}\left\{ \begin{pmatrix} 2xy \\ 0 \end{pmatrix}, \begin{pmatrix} -y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x^2 \\ -2xy \end{pmatrix}, \begin{pmatrix} xy \\ -y^2 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \end{pmatrix} \right\}. \tag{2.2.38}$$

Clearly, from this set, the vectors

$$\begin{pmatrix} -2xy \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x^2 \\ -2xy \end{pmatrix}, \begin{pmatrix} xy \\ -y^2 \end{pmatrix} \tag{2.2.39}$$

are linearly independent and, hence, second-order terms that are linear combinations of these four vectors can be eliminated. To determine the nature of the second-order terms that cannot be eliminated (i.e., $F_2^r(y)$), we must compute a space complementary to $L_J(H_2)$. This space, denoted G_2, will be two dimensional.

In computing G_2 it will be useful to first obtain a matrix representation for the linear operator $L_J(\cdot)$. This is done with respect to the basis given in (2.2.36) by constructing the columns of the matrix from the coefficients multiplying each basis element that are obtained when $L_J(\cdot)$ acts individually on each basis element of H_2 given in (2.2.36). Using (2.2.37), the matrix representation of $L_J(\cdot)$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2.2.40}$$

One way of finding a complementary space G_2 would be to find two "6-vectors" that are linearly independent and orthogonal (using the standard inner product in $\mathbb{R}^6$) to each column of the matrix (2.2.40), or, in other words, two linearly independent left eigenvectors of zero for (2.2.40). Due to the fact that most entries of (2.2.40) are zero, this is an easy calculation, and two such vectors are found to be

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{2.2.41}$$

Hence, the vectors

$$\begin{pmatrix} x^2 \\ \frac{1}{2}xy \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \end{pmatrix} \tag{2.2.42}$$

span a two-dimensional subspace of H_2 that is complementary to $L_J(H_2)$. This implies that the normal form through second-order is given by

$$\begin{aligned} \dot{x} &= y + a_1 x^2 + \mathcal{O}(3), \\ \dot{y} &= a_2 xy + a_3 x^2 + \mathcal{O}(3), \end{aligned} \tag{2.2.43}$$

where a_1, a_2, and a_3 represent constants.

Now our choice of G_2 is certainly not unique. Another choice might be

$$G_2 = \text{span}\left\{ \begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \end{pmatrix} \right\}. \tag{2.2.44}$$

This complementary space can be obtained by taking the vector

$$\begin{pmatrix} x^2 \\ \frac{1}{2}xy \end{pmatrix} \tag{2.2.45}$$

given in (2.2.42) and subtracting from it the vector

$$\begin{pmatrix} 0 \\ -\frac{1}{2}xy \end{pmatrix} \tag{2.2.46}$$

contained in $L_J(H_2)$. This gives the vector

$$\begin{pmatrix} x^2 \\ 0 \end{pmatrix}. \tag{2.2.47}$$

For the other basis element of the complementary space, we simply retain the vector

$$\begin{pmatrix} 0 \\ x^2 \end{pmatrix} \tag{2.2.48}$$

given in (2.2.42). With this choice of G_2 the normal form becomes

$$\begin{aligned} \dot{x} &= y + a_1 x^2 + \mathcal{O}(3), \\ \dot{y} &= a_2 x^2 + \mathcal{O}(3). \end{aligned} \tag{2.2.49}$$

This normal form near a fixed point of a planar vector field with linear part given by (2.2.35) was first studied by Takens [1974].

Another possibility for G_2 is given by

$$G_2 = \text{span}\left\{ \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \end{pmatrix} \right\}, \tag{2.2.50}$$

where these two vectors are obtained by adding the appropriate linear combinations of vectors in $L_J(H_2)$ to the vectors given in (2.2.38). With this choice of G_2 the normal form becomes

$$\begin{aligned} \dot{x} &= y + \mathcal{O}(3), \\ \dot{y} &= a_1 x^2 + b_2 xy + \mathcal{O}(3); \end{aligned} \tag{2.2.51}$$

this is the normal form for a vector field on $\mathbb{R}^2$ near a fixed point with linear part given by (2.2.35) that was first studied by Bogdanov [1975].