M638 CONTINUOUS DYNAMICAL SYSTEMS

Midterm Exam – Spring 2017

Name: ____

Instructions

1. This midterm exam is considered, strictly, individual: all work should be your own, carried out individually, not in a group.
2. Show all analytical work.
3. Graphical and/or computational work can also be included in support of the analysis. But unless otherwise stated in the problem, answers based purely on graphical and/or computational work are considered incomplete.

I, ____________________, pledge that this exam is completely my own work, and that I did not consult, take, borrow or steal any portions from any other person. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

All work that you complete in this class should be your own. In particular, exams, homework, and quizzes, are considered individual work. Copying someone else work, or allowing to be copied, is considered cheating and will result in an "F" in the course.

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Signature

1. (25 Pts.) Population Dynamics

Imagine a single cow in a single field of modest size. If the cow were introduced to the field after the field had lain fallow for a time, there would be ample vegetation. The cow would be well nourished and the vegetation would continue to grow. Suppose, however, that the same cow was placed in the same field after a herd of cows had grazed for several weeks. There would be little vegetation and the grazing cow might eat new grass blades as they appeared. Neither the cow nor the field would flourish. May developed a theoretical model for the dynamics of the amount of vegetation V:

$$\frac{dV}{dt} = G(V) - Hc(V), \qquad G(V) = rV(1 - V/k), \qquad c(V) = \frac{\beta V^2}{V_0^2 + V^2} \tag{1}$$

where $G(V)$ describes the growth of vegetation, $c(V)$ is the consumption of vegetation per cow, H is the number of cows in the herd, r, k, β and V_0 are positive constants.

(a) Consider three cases: $H = 10$, $H = 20$, and $H = 30$ cows. Let $r = 1/3$, $k = 25$, $\beta = 0.1$ and $V_0 = 3$. In one single graph, plot $Hc(V)$ and $G(V)$ as functions of V. Use the graph to identify how the number of equilibrium points changes as H varies.

(b) Analytically, calculate *all* equilibrium points and study their stability.

(c) Sketch a phase-line diagram indicating the stability of each equilibrium point for each individual case.

(d) Write a brief explanation of the results.

2. (25 Pts.) Laser Dynamics

Milonni and Eberly (1988) show that after certain reasonable approximations, quantum mechanics leads to the following model of a laser

$$\begin{cases} \dfrac{dn}{dt} &= GnN - kn \\ \dfrac{dN}{dt} &= -GnN - fN + p \end{cases} \tag{2}$$

where G is the gain coefficient for stimulated emission, k is the decay rate due to loss of photons by mirror transmission, f is the decay rate for spontaneous emission, and p is the pump strength. All parameters are positive, except p, which can have either sign.

(a) Suppose that N relaxes much more rapidly than n. Then we may make the quasi-static approximation $\dot{N} = 0$. Given this approximation, express $N(t)$ in terms of $n(t)$ and derive a first-order system for n.

(b) Show that $n^* = 0$ becomes unstable for $p > p_c$, where p_c is to be determined.

(c) What type of bifurcation occurs at the laser threshold p_c ?

(d) For what range of parameters is it valid to make the approximation?

3. (25 Pts.) Duffing Oscillator
Consider a Duffing oscillator of the form

$$\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + \lambda x - x^3 = 0 \tag{3}$$

where μ and λ are constant parameters.

(a) Convert the second order ODE (3) to a first-order system of ODE's.

(b) Analytically, calculate *all* equilibrium points and study their stability.

(c) Find the curves in (μ, λ) parameter-space at which the eigenvalues of the linearized Jacobian matrix, J, are purely imaginary, which is equivalent to the condition: $\text{trace}(J) = 0$ and $\det(J) > 0$. This locus of points is called a Hopf bifurcation.

(d) Sketch a diagram in the (μ, λ) plane illustrating the change in stability of each equilibrium point when both μ and λ change. Use pplane or any other equivalent software to sketch phase portraits of different types of behaviors.

4. (25 Pts.) Two-Species Trimolecular Reactions
Schnackenberg (1979) considered a class of two-species simplest, but chemically plausible, trimolecular reactions which can admit periodic solutions. After using the Law of Mass Action and nondimensionalizing, Schnackenberg reduced the system to

$$\begin{cases} \dfrac{dx}{dt} = a - x + x^2 y \\ \dfrac{dy}{dt} = b - x^2 y \end{cases} \tag{4}$$

where $a > 0$, $b > 0$ are parameters and $x > 0$, $y > 0$ are dimensionless concentrations.

(a) Show that all trajectories eventually enter a certain trapping region, to be determined. Make the trapping region as small as possible. (Hint: Examine the ratio $\dot{y}/\dot{x}$ for large x.

(b) Show that the system has a unique fixed point, and classify it through the linearization process.

(c) Show that the system undergoes a Hopf bifurcation when $b - a = (a + b)^3$.

(d) Is the Hopf bifurcation subcritical or supercritical? Use a computer to decide.

(e) Plot the stability diagram in (a, b) parameter space. Hint: It is a bit confusing to plot the curve $b - a = (a + b)^3$, since it requires analyzing a cubic. Show that the bifurcation curve can be expressed in parametric form $a = \frac{1}{2}x_E(1 - x_E^2)$, $b = \frac{1}{2}x_E(1 + x_E^2)$, where $x_E > 0$ is the x-coordinate of the fixed point. Then plot the bifurcation curve from these parametric equations.